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Abstract

Full Text

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SOME RIGIDITY CRITERIA FOR SURFACES OF REVOLUTION

(Presented by Academician S. L. Sobolev on 23 IV 1957)

In the works ^(1,2), I. N. Vekua proved the following theorem:

An ovaloid with one planar opening, in ideal contact with an orthogonal ideally smooth and absolutely rigid surface, is rigid with respect to infinitesimal bendings.

Recently B. Boyarskii considered the case of a nonorthogonal constraint for the sphere ⁽²⁾. The aim of the present paper is to indicate some rigidity criteria in the case of a nonorthogonal constraint for an ovaloid of revolution with one planar opening. The paper also indicates a rigidity criterion for a certain class of surfaces of revolution of negative curvature.

1. We denote by $\mathbf{r}(z, \vartheta)$

$$\mathbf{r}(z, \vartheta) = z\mathbf{k} + \rho(z)\mathbf{e}(\vartheta) \quad (1)$$

the surface of revolution (S), where $\rho(z)$ is the meridian of the surface; \mathbf{k} is the unit vector of the axis of revolution; $\mathbf{e}(\vartheta)$ is the unit vector perpendicular to \mathbf{k} , which, as ϑ varies, describes a circle with axis \mathbf{k} and arc length ϑ . Let $\mathbf{g}(\vartheta) = \mathbf{k} \times \mathbf{e}(\vartheta)$; $\mathbf{k}, \mathbf{e}(\vartheta)$ are a right-handed trihedron.

Further, let \mathbf{U} be the bending vector of the surface, which is expanded in the basis $\mathbf{e}, \mathbf{g}, \mathbf{k}$ in the form

$$\mathbf{U}(z, \vartheta) = u(z, \vartheta)\mathbf{e}(\vartheta) + v(z, \vartheta)\mathbf{g}(\vartheta) + w(z, \vartheta)\mathbf{k}. \quad (2)$$

Since an infinitesimal bending is being considered, the vector \mathbf{U} will satisfy the equation

$$d\mathbf{r} d\mathbf{U} = 0 \quad (3)$$

or the system of equations

$$\rho'(z)u_z + w_z = 0; \quad (4a)$$

$$v_\vartheta + u = 0; \quad (4b)$$

$$\rho(z)v_z + \rho'(z)(u_\vartheta - v) + w_\vartheta = 0. \quad (4c)$$

Obviously, equation (3) always admits a solution of the form

$$\mathbf{U} = \vec{\Omega} \times \mathbf{r} + \mathbf{C}, \quad (5)$$

where $\vec{\Omega}$ and \mathbf{C} are arbitrary constant vectors. This vector causes no intrinsic deformation of the surface. Therefore vectors of the form (5) will be called *trivial solutions* of equation (3), or of system (4).

Eliminating $u(z, \vartheta)$ and $w(z, \vartheta)$ from system (4), we obtain

$$\rho v_{zz} - \rho'' v_{\vartheta\vartheta} - \rho'' v = 0. \quad (6)$$

For a surface of revolution the function $v(z, \vartheta)$ must be periodic—function of ϑ with period 2π . Therefore the function $v(z, \vartheta)$ can be expanded in a Fourier series in ϑ :

$$v(z, \vartheta) = \sum_{k=0}^{\infty} [a_k(z) \cos k\vartheta + b_k(z) \sin k\vartheta]. \quad (7)$$

In view of (6), the coefficients $a_k(z), b_k(z)$ must satisfy the equation

$$\rho(z)\psi_k''(z) + (k^2 - 1)\rho''(z)\psi_k(z) = 0. \quad (8)$$

If equation (8), for $k \geq 2$, admits no bounded, twice differentiable solutions in the interval of variation $z : 0 \leq z \leq z_1$, then equation (6), and consequently system (4), admit no nontrivial bounded solutions (3). In this case the surface (S) is called **rigid with respect to infinitesimally small bendings**.

2. Let (S) be a surface of revolution of positive curvature, obtained by rotating a convex smooth curve $\widehat{OA} : \rho = \rho(z)$ ($0 \leq z \leq z_1$) (Fig. 1).

Let L be the contour of the surface (S), i.e. the circle of radius $\rho(z_1)$, lying in the plane $z = z_1$ with center on the z -axis.

Suppose that into the opening L there is inserted some rigid cone of revolution, which will offer no resistance to displacements of the points of the surface contour along the cone. Consequently, constraints of this kind fix at each point of the surface contour only one degree of freedom, leaving the other two completely unrestricted. Such a contour constraint gives for system (4) a boundary condition of the form

$$w(z, \vartheta) - u(z, \vartheta) \operatorname{ctg} \vartheta_1 = 0 \quad (\text{on } L), \quad (9)$$

Fig. 1

Figure 1: Fig. 1

where θ is the angle formed by the generator of the cone Λ and the z -axis.

Denote by Λ_0, Λ_1 the cones whose generators are perpendicular at the point A to the curve $\rho(z)$ and to the straight line \overline{OA} , respectively (Fig. 1). If ϑ_0, ϑ_1 are the angles they form with the z -axis ($\vartheta_1 > \vartheta_0$), then the following theorem holds.

Theorem. *Under a contour constraint of the form (9), the surface (S) will be rigid if*

$$0 \leq \theta \vee \theta_0, \quad \theta_1 \leq \theta \leq \pi, \quad (10)$$

whereas there exists an infinite sequence of angles $\{\theta_k\}$ ($k \geq 2$), $\theta_0 < \theta_k < \theta_1$, tending to θ_0 , for which system (4) will have nontrivial solutions, i.e. the surface will not be rigid.

Proof. In view of (9), (4), (4), and (7), the boundary condition for equation (8) can be written in the form

$$\rho\psi'_k + \rho'(k^2 - 1)\psi_k + k^2\psi_k \operatorname{ctg} \theta = 0 \quad \text{on } L \quad (k \geq 2). \quad (11)$$

Taking into account that we are interested only in a bounded solution of equation (8), it is easy to obtain that the functions $\psi_k(z)$ ($k \geq 2$) have the following properties: 1) $\psi_k(z) \geq 0$, $\psi''_k(z) > 0$, $0 \leq z \leq z_1$; 2) $\psi_k(z)/\rho(z)$ is an increasing function of z , with $\lim_{z \rightarrow 0} \psi_k(z)/\rho(z) = 0$; 3) $\rho\psi'_k - \rho'\psi_k = 0$ for $z = 0$. Therefore the inequalities hold

$$-\frac{\rho(z_1)}{z_1} < \operatorname{ctg} \theta = -\frac{\rho(z_1)\psi'_k(z_1) - \rho'(z_1)\psi_k(z_1)}{k^2\psi_k(z_1)} - \rho'(z_1) < -\rho'(z_1). \quad (12)$$

Our theorem follows directly from this.

Example 1. Consider an ellipsoid of revolution (S), whose meridian has the form

$$\rho(z) = C\sqrt{R^2 - (z - R)^2}, \quad 0 \leq z \leq z_1,$$

where $C > 0$, $R > 0$ are arbitrary constants.

The following assertion holds:

If the generators of the cone serving as the contour constraint for the surface (S) make with the z -axis the angle θ_k :

$$\theta_k = \text{ctg}^{-1} \left(-C \frac{R - k(z_1 - R)}{k\sqrt{R^2 - (z_1 - R)^2}} \right) \quad (k \geq 2), \quad (13)$$

then the surface (S) will not be rigid. In all other cases the surface (S) will be rigid.

It follows immediately from this that, for $z_1 = R + R/k$, $k = 2, 3, \dots$, the surface (S) will admit a sliding along the plane $z = z_1$. In the particular case when $C \equiv 1$, this fact was proved by other methods first by Rembs ⁽⁴⁾ and then, recently, by B. Boyarskii ⁽²⁾.

Example 2. Consider a surface of revolution (S), whose meridian has the form $\rho(z) = Cz^\alpha$, $0 \leq z \leq z_1$, where $0 < \alpha < 1$, $C > 0$ is an arbitrary constant. The following assertion holds:

If the generators of the cone make with the z -axis the angle θ_k :

$$\theta_k = \text{ctg}^{-1} \left(-C \frac{1 + 2\alpha(k^2 - 1) + \sqrt{4(k^2 - 1)\alpha(1 - \alpha) + 1}}{2k^2 z_1^{1-\alpha}} \right) \quad (k \geq 2), \quad (14)$$

then the surface (S) will not be rigid. In all other cases (S) will be rigid.

3. We proceed to the consideration of a surface of revolution of negative curvature. It is known that if the condition $U = 0$ (i.e. $u \equiv v \equiv w \equiv 0$) is prescribed on one contour of the surface, then the surface will be rigid. If on each contour of the surface only one of the conditions of the form $u = 0$, $v = 0$, $w = 0$, $du + bw = 0$ is prescribed, then the surface will no longer always be rigid, and, moreover, the rigidity of the surface will be unstable. We give an example explaining this fact, and indicate an additional condition for stable rigidity of the surface.

Consider a one-sheeted hyperboloid of revolution (S), whose meridian has the form $\rho = C\sqrt{z^2 + b^2}$, $0 \leq z \leq z_1$, where $C > 0$, b are arbitrary constants; L_0, L_1 are the contours of (S) lying in the planes $z = 0$, $z = z_1$, respectively.

Let the boundary conditions for the surface (S) be the following:

$$w(z, \vartheta) = 0 \quad \text{on } L_0, \quad v(z, \vartheta) = 0 \quad \text{on } L_1. \quad (15)$$

Then the following assertion holds:

If

$$z_1 \neq b \operatorname{tg} \frac{2m+1}{2k} \pi \quad (m = 0, 1, \dots; k = 1, 2, \dots),$$

then the surface (S) will be rigid*.

If

$$z_1 = b \operatorname{tg} \frac{2m_0+1}{2k_0} \pi, \quad (k_0, 2m_0+1) = 1,$$

then equation (6) will admit nontrivial solutions

$$\begin{aligned} v(z, \vartheta) = & \left[\alpha_{(2n+1)k_0} \cos(2n+1)k_0\vartheta + \beta_{(2n+1)k_0} \sin(2n+1)k_0\vartheta \right] \times \\ & \times \sqrt{b^2 + z^2} \cos \left[(2n+1)k_0 \operatorname{tg}^{-1} \left(\frac{z}{b} \right) \right], \end{aligned} \quad (16)$$

where $\alpha_{(2n+1)k_0}$, $\beta_{(2n+1)k_0}$ ($n = 0, 1, 2, \dots$) are arbitrary real numbers.

* Obviously, the rigidity of the surface (S) is unstable, since the set of points $\left\{ \operatorname{tg} \frac{2m+1}{2k} \pi \right\}$ ($m = 0, 1, \dots; k = 1, 2, \dots$) is dense on the real axis.

If the additional condition is imposed

$$w(z, \vartheta) = 0 \quad \text{for } z = z_1, \quad 0 \leq \vartheta \leq \vartheta_1, \quad (17)$$

where

$$\vartheta_1 = \frac{2 \operatorname{tg}^{-1}(z_1/b)}{2m_0+1}, \quad (18)$$

then from (16) we obtain

$$v(z, \vartheta) \equiv 0.$$

This means that, in order for the surface to be stably rigid, it is sufficient, in the presence of conditions (15), also to fix some arc σ on the contour L_1 , the length of this arc being not less than the quantity $2 \operatorname{tg}^{-1}(z_1/b)$, which is the distance along the contour L_1 between two asymptotic lines issuing from one point of the contour L_0 . (We note that if only this condition is imposed, then it ensures rigidity only of the part of the surface bounded by the arc σ and by two asymptotic lines issuing from the ends of the arc σ .)

An analogous result can be obtained under the boundary conditions

$$v(z, \vartheta) = 0 \quad \text{on } L_0 + L_1, \quad w(z, \vartheta) = 0 \quad \text{for } z = z_1, \quad 0 \leq \vartheta \leq 2 \operatorname{tg}^{-1} \frac{z_1}{b}$$

and so on.

In conclusion I express my gratitude to I. N. Vekua, under whose supervision this work was carried out.

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CITED LITERATURE

¹ I. N. Vekua, *Czechoslovak Mathematical Journal*, **6** (81), 143 (1956). ² I. N. Vekua, *Theory of Generalized Analytic Functions and Its Applications in Mechanics and Geometry* (in press). ³ S. Cohn-Vossen, *Uspekhi Mat. Nauk*, **9**, no. 1, 63 (1954). ⁴ E. Rembs, *Math. Ann.*, **111**, 587 (1935).

Note: Figure translations are in progress. See original paper for figures.

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