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Abstract

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HOMOLOGY GROUPS OF NILPOTENT ALGEBRAS

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In this note we study the upper homology groups $H^n(N, k)$, where N is a nilpotent associative algebra of finite rank over an arbitrary field k, and as the N-operator module of coefficients we take the field k itself, on which the elements of N act trivially. All properties of homology groups needed below can be found in the book ([1]).

It is known that the group $H^n(N, k)$ does not change if the algebra N is replaced by the algebra A obtained from N by formal adjoining of an identity. Thus one obtains any algebra whose quotient algebra by the radical is isomorphic to the ground field. This property is possessed, in particular, by the group algebra A of a finite p-group G over the field of p elements. In this case the group $H^n(A, k)$ coincides with the homology group $H^n(G, Z_p)$ of the group G with coefficients in the cyclic group of order p.

The group $H^n(N, k)$ is a vector space of finite dimension over the field k. The dimension of this space is denoted by b_n and will be called the n-dimensional Betti number of the algebra N. Analogous notation will be used for the groups $H^n(G, Z_p)$. The study of Betti numbers is the main aim of the present work. The formulation of the results obtained in this direction becomes more natural if one considers the Poincaré function associated with the algebra N,

$$R_N(t) = \sum_{n=0}^{\infty} b_n t^n;$$

the function $R_G(t)$ is defined analogously, where G is a finite p-group.

We have obtained the following results:

Theorem 1. Let $N = N_1 + \cdots + N_m$ be a direct sum of m nilpotent algebras; let $R_N(t) = R(t)$ and $R_{N_i}(t) = R_i(t)$ be the corresponding Poincaré functions. Then

$$\frac{1}{R(t)} - 1 = \sum_{i=1}^{m} \left(\frac{1}{R_i(t)} - 1 \right).$$

Theorem 2. The Betti numbers of a nilpotent algebra N are connected by the relations

$$b_n - b_{n-1} + \dots + (-1)^n b_0 \geq \frac{1 + (-1)^n}{2}, \quad n = 1, 2, \dots, \tag{1}$$

which may be written in the form

$$\frac{1}{1+t} R_N(t) \gg \frac{1}{1-t^2}. \tag{2}$$

Here the relation $F(t) \gg G(t)$ means that each coefficient of the power series $G(t)$ does not exceed the corresponding coefficient of $F(t)$.

In particular, for $n = 2$ relation (1) gives $b_2 \geqslant b_1$. For the case of p-groups this shows that the number of inequivalent extensions of the cyclic group Z_p by means of the p-group G is not less than p^d, where d is the number of generators of G.

Multiplying relation (2) by $1 + t$, we obtain

$$R_N(t) \gg \frac{1}{1-t},$$

whence Theorem 3 follows.

Theorem 3. The Betti numbers of a nilpotent algebra and of a finite p-group are positive.

An upper estimate for the Betti numbers is given by Theorem 4.

Theorem 4. For the function $R_N(t)$ corresponding to a nilpotent algebra N of rank r, the inequality

$$R_N(t) \ll \frac{1}{1-rt}$$

holds.

For the function $R_G(t)$ corresponding to a group G of order p^ν, the inequality

$$R_G(t) \ll \frac{1}{(1-t)^\nu} \tag{3}$$

holds.

Corollary. The radius of convergence ρ of the series $R_N(t)$ lies within the bounds

$$\frac{1}{r} \leqslant \rho \leqslant 1.$$

The radius of convergence of the series $R_G(t)$ is equal to 1.

Let us give several examples.

1. For a cyclic group G

$$R_G(t) = \frac{1}{1-t}.$$

Since, by the known Künneth relations, for the direct product $G_1 \times G_2$ of groups G_1 and G_2

$$R_{G_1 \times G_2}(t) = R_{G_1}(t) \cdot R_{G_2}(t),$$

it follows that for an abelian group G with d generators

$$R_G(t) = \frac{1}{(1-t)^d}.$$

2. The group G is a semidirect extension of a cyclic group by a cyclic group:

$$G = \{x, y\}, \qquad x^{p^\alpha} = y^{p^\beta} = 1, \qquad x^{-1}yx = y^{1+u}, \qquad (1+u)^{p^\alpha} - 1 = mp^\beta;$$

a) $m \equiv 0 \pmod{p}$;

$$R_G(t) = \frac{1}{(1-t)^2};$$

b) $(m, p) = 1$;

$$R_G(t) = \frac{1}{(1-t)^2(1+t^2+t^4+...+t^{2(p-1)})} = \frac{1+t}{(1-t)(1-t^{2p})}.$$

3. N is the quotient algebra of the algebra of polynomials without constant term in d variables by the ideal of polynomials of degree $\geqslant 3$,

$$R_N(t) = \frac{(1+t)^d}{1 - \sum_1^d \binom{i+1}{2}\binom{d+2}{i+2} t^{i+1}}.$$

4. N is the quotient algebra of the algebra of polynomials without constant term in two variables by the ideal of polynomials of degree $\geqslant r$,

$$R_N(t) = \frac{1+t}{1-t-rt^2}.$$

5. N is the reduced free noncommutative algebra with the relation $N^r = 0$ and d generators,

$$R_N(t) = \frac{1+dt}{1-d^r t^2}.$$

These and many other examples considered by us make it possible to state the hypothesis that, for any nilpotent algebra of finite rank, the function $R_N(t)$ is a rational function of t. In the case of a finite p-group G, all poles of the function $R_G(t)$ would then be roots of unity.

The following theorem also speaks in favor of this hypothesis.

Theorem 5. *If all Betti numbers of a nilpotent algebra N over a finite field k are bounded in the aggregate, then $R_N(t)$ is a rational function.*

The following example shows that there exist infinitely many algebras with bounded Betti numbers.

N has two generators e_1 and e_2; $N^5 = 0$; k is an arbitrary field of characteristic $\neq 2$, and

$$e_2 e_1 = -e_1 e_2 + a e_1^3 + b e_1^2 e_2 + c e_1^4; \qquad e_2^2 = f e_1^2 + g e_1^2 e_2 + h e_1^4;$$

$$e_1^3 e_2 = 0; \qquad b \cdot f \left(f + \left(\frac{g-a}{b} \right)^2 \right) \neq 0;$$

$$R_N(t) = \frac{1+2t+2t^2+t^3}{1-t^4} = \frac{1+t+t^2}{1-t+t^2-t^3}.$$

As is known, the generalized quaternion group has the same Poincaré function.

The proofs of the formulated theorems follow easily from consideration of the special complete resolution

$$0 \xleftarrow{\partial_{-1}} X_{-1} \xleftarrow{\partial_0} X_0 \leftarrow \cdots \leftarrow X_{n-1} \xleftarrow{\partial_n} X_n \leftarrow \cdots, \tag{4}$$

in which

$$X_{-1} = k, \qquad X_0 = A = \alpha e + N, \quad \alpha \in k, \quad e^2 = e, \quad e\nu = \nu, \quad \partial_0(\alpha e + \nu) = \alpha \ (\nu \in N).$$

Putting

$$\mathrm{Ker}(\partial_n) = J_n,$$

we denote by b_{n+1} the minimal number of generators of J_n as an A-operator module. As is known, b_{n+1} is equal to the dimension of the linear space $J_n - NJ_n$. We put

$$X_{n+1} = E^{b_{n+1}} \otimes A,$$

where $E^{b_{n+1}}$ is a linear space over k of dimension b_{n+1}, and $\otimes$ denotes tensor product. It is easy to see that the numbers b_n coincide with the Betti numbers of the algebra N. In the proof of Theorem 1 one may, of course, restrict oneself to the case $m = 2$. Let $N = N_1 + N_2$, $R_1(t) = \sum a_n t^n$, $R_2(t) = \sum b_n t^n$, $R(t) = \sum c_n t^n$; A_n, B_n, and C_n are the kernels J_n of the corresponding exact sequences of type (4). Obviously,

$$C_0 = A_0 + B_0,$$

$$C_1 = A_1 + B_1 + E^{a_1} \otimes B_0 + E^{b_1} \otimes A_0 = A_1 + B_1 + E^{a_1} \otimes C_0 + E^{b_1 - a_1} \otimes A_0.$$

By induction, the following relations easily follow:

$$C_{n-1} = A_{n-1} + B_{n-1} + \sum_1^{n-1} E^{d_i} \otimes C_{n-1-i} + \sum_1^{n-1} E^{b_i - d_i} \otimes A_{n-1-i}, \qquad (5)$$

where

$$d_n = \sum_0^{n-1} (b_i - d_i) a_{n-i} \quad (d_0 = 0). \qquad (6)$$

From (5) and (6) it follows that

$$c_n = b_n + \sum_0^n d_{n-i} c_i, \qquad b_n + \sum_0^n d_i a_{n-i} - \sum_0^n b_i a_{n-i} = 0,$$

i.e.

$$R(t) = R_2(t) + R(t) D(t), \qquad (7)$$

$$R_2(t) + R_1(t)D(t) - R_1(t)R_2(t) = 0, \tag{8}$$

where

$$D(t) = \sum d_n t^n.$$

Theorem 1 easily follows from (7) and (8).

Theorem 2 follows from the fact that, in the sequence (4), the ranks over the field k of the modules X_n and J_n are connected by the relations

$$r(X_n) = b_n(1 + r(N)) = r(J_n) + r(J_{n-1}).$$

By induction from this we obtain that

$$r(J_{n-1}) = (b_{n-1} - b_{n-2} + ... + (-1)^{n-1}b_0)(1 + r(N)) + (-1)^n,$$

whence, in view of the fact that

$$b_n \geq \frac{r(J_{n-1})}{1 + r(N)},$$

Theorem 2 easily follows.

Theorems 3, 4, and 5 follow in an analogous manner from consideration of the sequence (4). The only exception is inequality (3) of Theorem 4. It follows from the fact that the Betti numbers of a central extension are majorized by the Betti numbers of the direct product. This fact is derived in the standard way from consideration of the Hochschild–Serre spectral sequence.

Remark added in proof. After the manuscript had already been submitted for publication, the authors learned that B. B. Venkov is engaged with an analogous problem. His results concerning p-groups partially coincide with ours.

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Note: Figure translations are in progress. See original paper for figures.