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Abstract

Full Text

PHYSICS

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ON THE STEADY-STATE TEMPERATURE OF A FLAT PLATE WASHED BY A REACTING GAS MIXTURE

(Presented by Academician N. N. Bogolyubov, 13 XII 1956)

The problem is considered of the steady-state temperature of a flat plate washed by a laminar gas flow in which a reaction of the type $X_2 \rightarrow 2X$ takes place. The temperature sought will be determined by solving the system of boundary-layer equations. The complete system of boundary-layer equations in the case under consideration may be written in the form:

the diffusion equation for the initial reaction product

$$\rho \left(u_1 \frac{\partial c_1}{\partial x_1} + u_2 \frac{\partial c_1}{\partial x_2} \right) = \frac{\partial}{\partial x_2} \left(\rho D \frac{\partial c_1}{\partial x_2} \right) - \rho Z_1; \quad (1a)$$

the continuity equation

$$\frac{\partial(\rho u_1)}{\partial x_1} + \frac{\partial(\rho u_2)}{\partial x_2} = 0; \quad (1)$$

the equation of conservation of momentum

$$\rho \left(u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_1}{\partial x_2} \right) = \frac{\partial}{\partial x_2} \left(\eta \frac{\partial u_1}{\partial x_2} \right); \quad (1)$$

the equation of conservation of energy

$$\rho \left(u_1 \frac{\partial(c_p \theta)}{\partial x_1} + u_2 \frac{\partial(c_p \theta)}{\partial x_2} \right) = \frac{\partial}{\partial x_2} \left(\lambda \frac{\partial T}{\partial x_2} + \eta u_1 \frac{\partial u_1}{\partial x_2} + q \rho D \frac{\partial c_1}{\partial x_2} \right); \quad (1)$$

$$c_1 + c_2 = 1; \quad (1)$$

$$p = R\rho T, \quad (1)$$

where

$$\theta = T + \frac{1}{2} \frac{u_1^2}{c_p} + \frac{q}{c_p} c_1; \quad (2)$$

ρ is the density of the mixture; T is the temperature of the mixture; u_1, u_2 are the components of the flow velocity in the boundary layer; $c_1 = \rho_1/\rho$ is the concentration of the initial reaction product (ρ_1 is the density of the component X_2); c_2 is the concentration of the reaction products or, if an inert component c_3 is present in the flow, the sum of the concentrations of the reaction products and the inert component (such a device, for example, may be used if the molecular weight of the reaction products does not differ greatly from the molecular weight of the inert component ⁽⁵⁾); D is the diffusion coefficient of the component X_2 ; η is the viscosity coefficient; λ is the thermal-conductivity coefficient; ρZ_1 is the reaction rate; q is the heat release of the reaction; c_p is the heat capacity of the mixture at constant pressure; p is the pressure of the mixture.

If we assume that the thermal Prandtl number $\text{Pr} = c_p \eta / \lambda$, the diffusion Prandtl number $\text{Pr}_1 = \eta / \rho D$, and the heat capacity of the mixture c_p do not depend on temperature, the energy-conservation equation (1)

is transformed to the form

$$\rho \left(u_1 \frac{\partial \theta}{\partial x_1} + u_2 \frac{\partial \theta}{\partial x_2} \right) = \frac{1}{\text{Pr}} \frac{\partial}{\partial x_2} \left(\eta \frac{\partial \theta}{\partial x_2} \right) + \frac{\text{Pr} - 1}{\text{Pr}} \frac{\partial}{\partial x_2} \left[\eta \frac{\partial}{\partial x_2} \left(\frac{u_1^2}{2c_p} \right) \right] + \frac{1}{\text{Pr}} \left(\frac{\text{Pr}}{\text{Pr}_1} - 1 \right) \frac{q}{c_p} \frac{\partial}{\partial x_2} \left(\eta \frac{\partial c_1}{\partial x_2} \right). \quad (3)$$

We consider the general case with respect to the magnitude of the Prandtl numbers: $\text{Pr} \neq 1$, $\text{Pr}_1 \neq 1$, and $\text{Pr} \neq \text{Pr}_1$. For the dependence of viscosity on temperature we shall use the linear law ⁽¹⁾

$$\eta = \eta_0 \frac{T}{T_0}, \quad (4)$$

where η_0, T_0 are constants.

As Chapman and Rubesin ⁽²⁾ showed, the inaccuracy of the linear approximation for the dependence of viscosity on temperature can be somewhat corrected by allowing for the difference of the Prandtl number from unity.

Let us consider the case of a heterogeneous reaction proceeding in the diffusion region. Then, in the diffusion equation (1a), the term ρZ_1 drops out, and the energy conservation equation has the form

$$\rho \left(u_1 \frac{\partial \theta}{\partial x_1} + u_2 \frac{\partial \theta}{\partial x_2} \right) = \frac{1}{\text{Pr}} \frac{\partial}{\partial x_2} \left(\eta \frac{\partial \theta}{\partial x_2} \right) + \frac{\text{Pr} - 1}{\text{Pr}} \frac{\partial}{\partial x_2} \left[\eta \frac{\partial}{\partial x_2} \left(\frac{u_1^2}{2c_p} \right) \right], \quad (5)$$

where

$$\theta = T + \frac{1}{2} \frac{u_1^2}{c_p}.$$

With the aid of the Dorodnitsyn transformation ⁽¹⁾, which consists in introducing, in place of the coordinate x_1 , the quantity y_1 :

$$y_1 = \int_0^{x_1} \frac{\rho}{\rho_0} dx_1, \quad (6)$$

and, in place of the coordinate x_2 , the quantity y_2 :

$$y_2 = \int_0^{x_2} \frac{\rho}{\rho_0} dx_2 = \frac{p}{p_0} \int_0^{x_2} \frac{T_0}{T} dx_2, \quad (7)$$

the equations of system (1) and the energy conservation equation in the form (5) take the form:

$$v_1 \frac{\partial c_1}{\partial y_1} + v_2 \frac{\partial c_1}{\partial y_2} = D_0 \frac{\partial^2 c_1}{\partial y_2^2}; \quad (8a)$$

$$\frac{\partial v_1}{\partial y_1} + \frac{\partial v_2}{\partial y_2} = 0; \quad (8b)$$

$$v_1 \frac{\partial v_1}{\partial y_1} + v_2 \frac{\partial v_1}{\partial y_2} = \nu_0 \frac{\partial^2 v_1}{\partial y_2^2}; \quad (8c)$$

$$v_1 \frac{\partial \theta}{\partial y_1} + v_2 \frac{\partial \theta}{\partial y_2} = \frac{\nu_0}{\text{Pr}} \frac{\partial^2 \theta}{\partial y_2^2} + \frac{\text{Pr} - 1}{\text{Pr}} \nu_0 \frac{\partial^2}{\partial y_2^2} \left(\frac{v_1^2}{2c_p} \right), \quad (8d)$$

where $v_1 = u_1$, $v_2 = \frac{T_0}{T} u_2 + u_1 \frac{p_0}{p} \frac{\partial y_2}{\partial x_1}$, $D_0, \nu_0 = \eta_0 / \rho_0$ are constants.

Let us introduce the stream function ψ . Then, putting $v_1 = \frac{\partial \psi}{\partial y_2}$, $v_2 = -\frac{\partial \psi}{\partial y_1}$, we satisfy the continuity equation (8b).

We shall solve system (8) under the following boundary conditions:

$$\text{at } y_2 = 0, \quad v_1 = v_2 = 0 \quad \left(\lambda \frac{\partial T}{\partial y_2} \right)_0 = -q \left(\rho D \frac{\partial c_1}{\partial y_2} \right)_0, \quad c_1 = 0; \quad (9)$$

$$\text{at } y_2 = \infty, \quad v_1 = u_\infty, \quad T = T_\infty, \quad c_1 = c_{1\infty} \quad \left(\theta = \theta_\infty = T_\infty + \frac{u_\infty^2}{2c_p} \right). \quad (10)$$

We seek the solution of the system in the form

$$\psi = \sqrt{\nu_0 u_\infty y_1} f(\tau), \quad \theta = \theta(\tau), \quad c_1 = c_1(\tau), \quad (11)$$

where $\tau = \frac{1}{2} y_2 \sqrt{u_\infty / \nu_0 y_1}$.

Substituting (11) into system (8), we obtain:

$$c_1'' + \text{Pr}_1 f c_1' = 0; \quad (12)$$

$$f''' + f f'' = 0; \quad (13)$$

$$\theta'' + \text{Pr} f \theta' = (1 - \text{Pr}) \frac{u_\infty^2}{8c_p} \frac{d^2}{d\tau^2} (f'^2), \quad (14)$$

where the prime denotes differentiation with respect to τ .

The solution of equation (13) is well known ⁽¹⁾. The function f is the Blasius function.

The solution of (12) is readily found by two quadratures, and the concentration profile is determined by the expression

$$c_1 = c_{1\infty} \frac{\int_0^\tau (f'')^{\text{Pr}_1} d\tau}{\int_0^\infty (f'')^{\text{Pr}_1} d\tau}. \quad (15)$$

We shall seek the solution of equation (14) in the form of the sum of the general solution of the homogeneous equation

$$M'' + \text{Pr} f M' = 0 \quad (16)$$

under the boundary conditions:

$$\begin{aligned} \text{for } \tau = 0 \quad M' &= -\frac{q}{c_p \text{Pr}_1} \left(\frac{dc_1}{d\tau} \right)_0, \\ \text{for } \tau = \infty \quad M &= 0 \end{aligned} \quad (17)$$

and a particular solution $N(\tau)$ of the nonhomogeneous equation

$$N'' + \text{Pr} f N' = (1 - \text{Pr}) \frac{u_\infty^2}{8c_p} \frac{d^2}{d\tau^2} (f'^2) \quad (18)$$

under the boundary conditions:

$$\begin{aligned} \text{for } \tau = 0 & \quad N' = 0, \\ \text{for } \tau = \infty & \quad N = \theta_\infty. \end{aligned} \quad (19)$$

The required solution of equation (16) has the form

$$M(\tau) = \frac{q}{c_p} \frac{\text{Pr}}{\text{Pr}_1} \left(\frac{dc_1}{d\tau} \right)_0 \frac{\int_\tau^\infty (f'')^{\text{Pr}} d\tau}{[f''(0)]^{\text{Pr}}}, \quad (20)$$

where

$$\left(\frac{dc_1}{d\tau} \right)_0 = c_{1\infty} \frac{[f''(0)]^{\text{Pr}_1}}{\int_0^\infty (f'')^{\text{Pr}_1} d\tau}.$$

A particular solution of the nonhomogeneous equation (18) is found by the method of variation of constants and has the form ⁽¹⁾

$$N(\tau) = T_\infty + \frac{u_\infty^2}{2c_p} + (1 - \text{Pr}) \frac{u_\infty^2}{8c_p} \int_\infty^\tau (f'')^{\text{Pr}} \int_0^\tau (f'')^{-\text{Pr}} \frac{d^2}{d\tau^2} (f'^2) d\tau d\tau. \quad (21)$$

Taking into account that the general solution of equation (14) is $\theta = M(\tau) + N(\tau)$ and that $\theta = T + u_1^2/2c_p$, we shall have the following expression for the temperature profile:

$$\begin{aligned} T = T_\infty + \frac{u_\infty^2}{2c_p} [1 - a(\tau, \text{Pr})] - \frac{u_\infty^2}{8c_p} f'^2 + \\ + \frac{qc_{1\infty}}{c_p} \frac{\text{Pr}}{\text{Pr}_1} \frac{[f''(0)]^{\text{Pr}_1} \int_\tau^\infty (f'')^{\text{Pr}} d\tau}{[f''(0)]^{\text{Pr}} \int_0^\infty (f'')^{\text{Pr}_1} d\tau}, \end{aligned} \quad (22)$$

where

$$a(\tau, \text{Pr}) = \frac{1 - \text{Pr}}{4} \int_{\tau}^{\infty} (f'')^{\text{Pr}} \int_0^{\tau} (f'')^{-\text{Pr}} \frac{d^2}{d\tau^2} (f'^2) d\tau d\tau. \quad (23)$$

From expression (22), for $\tau = 0$ we obtain the desired formula for the steady temperature of a plane plate:

$$T_{\omega} = T_{\infty} + \frac{u_{\infty}^2}{2c_p} [1 - a(\text{Pr})] + \frac{qc_{1\infty}}{c_p} \frac{\text{Pr}}{\text{Pr}_1} \frac{b(\text{Pr}_1)}{b(\text{Pr})}; \quad (24)$$

where

$$a(\text{Pr}) = \frac{1 - \text{Pr}}{4} \int_0^{\infty} (f'')^{\text{Pr}} \int_0^{\tau} (f'')^{-\text{Pr}} \frac{d^2}{d\tau^2} (f'^2) d\tau d\tau; \quad (25)$$

$$b(\text{Pr}) = \frac{[f''(0)]^{\text{Pr}}}{\int_0^{\infty} (f'')^{\text{Pr}} d\tau}; \quad (26)$$

$$b_1(\text{Pr}) = \frac{[f''(0)]^{\text{Pr}_1}}{\int_0^{\infty} (f'')^{\text{Pr}_1} d\tau}. \quad (27)$$

These functions can, with a good degree of accuracy, be represented by the expressions ⁽³⁾

$$a(\text{Pr}) = 1 - \text{Pr}^{1/2}, \quad b(\text{Pr}) = \alpha \text{Pr}^{1/3}, \quad b_1(\text{Pr}) = \alpha \frac{\text{Pr}^{1/3}}{\text{Pr}_1}, \quad (28)$$

where α is the Blasius constant.

Substituting (28) into (24), we finally obtain:

$$T_{\omega} = T_{\infty} + \frac{u_{\infty}^2}{2c_p} \text{Pr}^{1/2} + \frac{qc_{1\infty}}{c_p} \left(\frac{\text{Pr}}{\text{Pr}_1} \right)^{2/3} \quad (29)$$

or, in dimensionless form,

$$\frac{T_{\omega} - T_{\infty}}{T_{\infty}} = \frac{\gamma - 1}{2} M^2 \text{Pr}^{1/2} + W \left(\frac{\text{Pr}}{\text{Pr}_1} \right)^{2/3}, \quad (30)$$

where M is the Mach number; $W = qc_{1\infty}/c_p T_{\infty}$ is a number equal to the ratio of the total chemical energy to the thermal energy of the flow far from the plate; γ is the ratio of heat capacities at constant pressure and at constant volume.

In the present work, by means of an exact solution of the system of differential equations for the boundary layer of reacting gas mixtures with variable physical properties, a formula has been obtained for the steady temperature of a plane plate, applicable at high velocities or when the heat of friction is comparable in magnitude with the heat released in the chemical reaction. In the case of small Mach numbers, expression (30) yields the result of D. A. Frank-Kamenetskii and N. Ya. Buben^(4,5), obtained from general considerations of the balance of heat fluxes at the surface.

In addition, concentration and temperature profiles have been found for a heterogeneous reaction proceeding in the diffusion region.

We note that in the work the influence of the thermodiffusion effect was not taken into account.

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Note: Figure translations are in progress. See original paper for figures.

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