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Abstract

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MATHEMATICS

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AN INVERSE PROBLEM FOR SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS CONTAINING A PARAMETER

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Consider the system of differential equations

$$\frac{dy_i}{dx} = \frac{i}{\lambda}\sum_{j=1}^{n} b_{ij}(x)y_j \quad (i = 1, 2, \dots, n;\ 0 \le x \le l), \tag{1}$$

where λ is a complex parameter, and denote by $W(x,\lambda)$ the fundamental matrix of solutions of this system:

$$\frac{dW(x,\lambda)}{dx} = \frac{i}{\lambda} b(x)W(x,\lambda), \quad W(0,\lambda) = I \quad (b(x) = \|b_{ij}(x)\|).$$

In the present article we establish certain sufficient conditions under which the coefficients $b_{ij}(x)$ of system (1) are uniquely recovered from the matrix $W(\lambda) = W(l,\lambda)$.

I. We shall say that a square matrix-function $W(\lambda)$ belongs to the class M^+ if it satisfies the following conditions: 1) the elements of the matrix $W(1/z)$ are entire functions of $z = 1/\lambda$, 2) $\lim_{\lambda\to\infty} W(\lambda) = I$, 3) $W(\lambda)W^*(\lambda) = I$ for $\operatorname{Im}\lambda = 0$, 4) $W(\lambda)W^*(\lambda) \ge I$ for $\operatorname{Im}\lambda > 0$. From conditions 1)–4) it follows that the expansion of the matrix-function $W(\lambda)$ in a series in negative powers of λ has the form $W(\lambda) = I + \frac{i}{\lambda}H + \cdots$, where H is a Hermitian nonnegative matrix. The trace of the matrix H we agree to call the **weight** of the matrix-function $W(\lambda)$. If $W(\lambda) = W_2(\lambda)W_1(\lambda)$ $(W_1(\lambda) \in M^+,\ W_2(\lambda) \in M^+)$, then the matrix $W_1(\lambda)$ will be called a **divisor** of the matrix-function $W(\lambda)$.

Consider a bounded linear operator A acting in a Hilbert space H. We shall assign the operator A to the class K^+ if: 1) the spectrum of the operator A consists of only the single point 0, 2) the space

$$E = \frac{A - A^*}{i}\, H$$

is finite-dimensional, and all eigenvalues ω_α $(\alpha = 1, 2, \dots, r)$ of the operator $\frac{A-A^*}{i}$ in E are positive, 3) the operator A is simple, i.e. H coincides with the closure of the linear span of vectors of the form $A^n e_\alpha$ $(n = 0, 1, 2, \dots;\ \alpha = 1, 2, \dots, r)$, where e_α is an orthonormal basis of eigenvectors of the operator $\frac{A-A^*}{i}$ in E $\left(\frac{A-A^*}{i} e_\alpha = \omega_\alpha e_\alpha\right)$. The number

$$l = \sum_{\alpha=0}^{r} \omega_\alpha$$

will be called the **non-Hermitian trace** of the operator A. Every matrix-function

$$W(\lambda) = I - i\Pi \|((A - \lambda E)^{-1} e_\alpha, e_\beta)\| \Pi^*,$$

where Π is an arbitrary square or rectangular matrix, satis-

satisfying the condition $\Pi^*\Pi = \Omega$,

$$\Omega = \begin{Vmatrix} \omega_1 & & \\ & \omega_2 & \\ & & \ddots\ \omega_r \end{Vmatrix},$$

is called **characteristic** [1,2] for the operator A. Let H_0 be some subspace in H, P_0 the projection operator onto H_0, and A_0 the operator acting in H_0 for which $A_0 f = P_0 A f$, $(f \in H_0)$. If $e_\alpha^{(0)}$ $(\alpha = 1, 2, \dots, r_0)$ is an orthonormal basis of eigenvectors of the operator A_0 in the subspace

$$E_0 = \frac{A_0 - A_0^*}{i} H_0,$$

then the matrix-function

$$W_0(\lambda) = I - i\Pi_0 \left\| ((A_0 - \lambda E)^{-1} e_\alpha^{(0)},\, e_\beta^{(0)}) \right\| \Pi_0^* \quad (\Pi_0 = \Pi U_0,\ U_0 = \|(e_\alpha, e_\beta^{(0)})\|)$$

is characteristic for the operator A_0 and is called a **projection** [2] of the matrix-function $W(\lambda)$ onto H_0.

Theorem 1. *In order that the matrix-function $W(\lambda)$ belong to the class M^+, it is necessary and sufficient that it be characteristic for some operator belonging to the class K^+.*

Theorem 2. *Let $W(\lambda)$ be the characteristic matrix-function of an operator $A \in K^+$ acting in the space H. In order that the matrix $W_1(\lambda)$ be a divisor of*

$W(\lambda)$, it is necessary and sufficient that it be a projection of $W(\lambda)$ onto some subspace $H_1 \subseteq H$, invariant with respect to A.

Every matrix-function $W(\lambda) \in M^+$ having weight l admits the multiplicative representation [3]

$$W(\lambda) = \int_0^l e^{\frac{i}{\lambda}\, dE(t)} = \lim_{\Delta t_i \to 0} \left(e^{\frac{i}{\lambda}\Delta E_p} \cdots e^{\frac{i}{\lambda}\Delta E_2} e^{\frac{i}{\lambda}\Delta E_1} \right), \tag{2}$$

where

$$E(t) = \int_0^t b(x)\, dx,$$

$b(x)$ is a certain Hermitian nonnegative summable matrix on $[0, l]$, for which $\operatorname{Sp} b(x) \equiv 1$, $0 = t_0 < t_1 < \cdots < t_p = l$, $\Delta E_k = E(t_k) - E(t_{k-1})$, $\Delta t_k = t_k - t_{k-1}$. From representation (2) it follows that $\|W(\lambda)\| < e^{l/|\lambda|}$. From the same representation it is seen that the matrix-function $W(\lambda)$ has the divisor

$$\int_0^{l_1} e^{\frac{i}{\lambda}\, dE(t)}$$

of any weight $l_1 < l$.

Theorem 3. *Let the matrix-function $W(\lambda) \in M^+$ have weight l. If for every $\varepsilon > 0$ there exists a sequence $\lambda_k \to 0$ for which $\|W(\lambda_k)\| > e^{(l-\varepsilon)/|\lambda_k|}$ $(k = 1, 2, 3, \ldots)$, then $W(\lambda)$ has one and only one divisor of the given weight $l_1 < l$.*

Proof. By virtue of Theorem 1, the matrix-function $W(\lambda)$ is characteristic for some operator $A \in K^+$. Let $W_1(\lambda)$ and $W_1'(\lambda)$ be divisors of the matrix-function $W(\lambda)$ having the common weight l_1. By virtue of Theorem 2, $W_1(\lambda)$ and $W_1'(\lambda)$ serve as projections of $W(\lambda)$ onto certain invariant subspaces H_1 and H_1' of the operator A. In [4] it is shown that, under the conditions of the theorem being proved, the operator A is unicellular. Consequently, one of the subspaces H_1, H_1' is part of the other. Since the operators A_1 and A_1', generated by the operator A respectively in H_1 and H_1', have a common non-Hermitian trace l_1, it follows that $H = H_1'$, and therefore $W_1(\lambda) = W_1'(\lambda)$.

II. The system of differential equations

$$\frac{dy_i}{dx} = \frac{i}{\lambda} \sum_{j=1}^{n} b_{ij}(x) y_j \quad (i = 1, 2, \ldots, n;\ 0 \le x \le l) \tag{3}$$

we shall call it **normalized** if the trace $\sum_{i=1}^{n} b_{ii}(x) \equiv 1$. If system (3) is not normalized, but the trace $\sum_{i=1}^{n} b_{ii}(x)$ differs from zero almost everywhere, then it can be normalized by making the change of independent variable

$$t = \int_0^x \sum_{i=1}^{n} b_{ii}(x)\, dx.$$

Theorem 4. *Suppose that on the segment $[0,l]$ there are given two normalized systems of differential equations*

$$\frac{dy_i}{dx} = \frac{i}{\lambda} \sum_{j=1}^{n} b_{ij}^{(1)}(x) y_j, \qquad \frac{dy_i}{dx} = \frac{i}{\lambda} \sum_{j=1}^{n} b_{ij}^{(2)}(x) y_j, \tag{4}$$

whose coefficient matrices $b^{(1)}(x) = \|b_{ij}^{(1)}(x)\|$ and $b^{(2)}(x) = \|b_{ij}^{(2)}(x)\|$ are Hermitian nonnegative, with the functions $b_{ij}^{(1)}(x)$ possessing absolutely continuous first derivatives and the rank of the matrix $b^{(1)}(x)$ equal to one for every $x \in [0,l]$. Denote by $W^{(1)}(x,\lambda)$ and $W^{(2)}(x,\lambda)$ the fundamental matrices of solutions of these systems. If $W^{(1)}(l,\lambda) \equiv W^{(2)}(l,\lambda)$, then $b_{ij}^{(1)}(x) \equiv b_{ij}^{(2)}(x)$.

Proof. Studying the asymptotics of the solutions of the first of systems (4), we find that the matrix-function $W^{(1)}(l,\lambda)$ satisfies the conditions of Theorem 3. Since

$$W^{(i)}(x,\lambda) = \int_0^x e^{\frac{i}{\lambda}}\, dE^{(i)}(x), \qquad E^{(i)}(x) = \int_0^x b^{(i)}(t)\, dt \quad (i = 1,2),$$

it follows that $W^{(1)}(x,\lambda)$ and $W^{(2)}(x,\lambda)$ are, for the matrix-function $W^{(1)}(l,\lambda)$, divisors of one and the same weight x. By virtue of Theorem 3, $W^{(1)}(x,\lambda) \equiv W^{(2)}(x,\lambda)$ for every fixed $x \in [0,l]$, and, consequently, $b^{(1)}(x) \equiv b^{(2)}(x)$.

Theorem 5. *Suppose that on the segment $[0,l]$ there are given normalized systems of differential equations (4), where $b^{(2)}(x)$ is a Hermitian nonnegative matrix with summable elements, and the matrix $b^{(1)}(x) = \xi^*(x)\xi(x)$, $\xi(x) = \|\xi_1(x) \dots \xi_n(x)\|$, where the vector $\xi(x)$ assumes on each of the intervals $[x_{i-1}, x_i]$ of some partition $0 = x_0 < x_1 < \dots < x_p = l$ a constant value $\xi^{(i)}$. If among the vectors $\xi^{(1)}, \xi^{(2)}, \dots, \xi^{(p)}$ there are no two neighboring ones that are mutually orthogonal, and $W^{(1)}(l,\lambda) \equiv W^{(2)}(l)$, then $b^{(1)}(x) \equiv b^{(2)}(x)$.*

Proof. Since $\xi^{(i)}\xi^{(i)*} = 1$ $(i = 1,2,\dots,p)$, the matrix-function

$$W^{(1)}(l,\lambda) = \exp\left[\frac{i}{\lambda}\xi^{(p)*}\xi^{(p)}\Delta x_p\right] \cdots \exp\left[\frac{i}{\lambda}\xi^{(2)*}\xi^{(2)}\Delta x_2\right] \exp\left[\frac{i}{\lambda}\xi^{(1)*}\xi^{(1)}\Delta x_1\right] =$$

$$= \left[I + \xi^{(p)*}\xi^{(p)}\left(e^{\frac{i}{\lambda}\Delta x_p} - 1\right)\right] \cdots$$

$$\cdots \left[I + \xi^{(2)*}\xi^{(2)}\left(e^{\frac{i}{\lambda}\Delta x_2} - 1\right)\right]\left[I + \xi^{(1)*}\xi^{(1)}\left(e^{\frac{i}{\lambda}\Delta x_1} - 1\right)\right]$$

again satisfies the conditions of Theorem 3, and the arguments given in Theorem 4 may be applied to it.

The following assertion is a consequence of the last theorem.

Let $A_1, A_2, \dots, A_n, B_1, B_2, \dots, B_m$ be Hermitian nonnegative matrices of the first rank, satisfying the conditions

$$s_p A_i = 1 \ (i = 1, 2, \dots, n), \qquad A_i \neq A_{i+1}, \quad A_i A_{i+1} \neq 0 \quad (i = 1, 2, \dots, n-1);$$

$$s_p B_j = 1 \ (j = 1, 2, \dots, m), \qquad B_j \neq B_{j+1} \quad (j = 1, 2, \dots, m-1),$$

$\alpha_1, \ \alpha_2, \dots, \alpha_n, \ \beta_1, \beta_2, \dots, \beta_m$ be certain positive numbers. If, for every z, the equality

$$e^{z\alpha_1 A_1} e^{z\alpha_2 A_2} \dots e^{z\alpha_n A_n} = e^{z\beta_1 B_1} e^{z\beta_2 B_2} \dots e^{z\beta_m B_m},$$

holds, then $n = m$, $\alpha_i = \beta_i$, and $A_i = B_i$ $(i = 1, 2, \dots, n)$.

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REFERENCES

[1] M. S. Livshits, *Matem. sborn.*, **34** (76), 1, 145 (1954).
[2] M. S. Brodskii, DAN, **77**, No. 5 (1950).
[3] V. P. Potapov, DAN, **72**, No. 5 (1950).
[4] M. S. Brodskii, DAN, **111**, No. 5 (1956).

Note: Figure translations are in progress. See original paper for figures.