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Abstract

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MATHEMATICS

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ON SYSTEMS OF TWO EQUATIONS OF BRIOT AND BOUQUET

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1. Consider a system of two differential equations (in general, in the complex domain)

$$x\frac{dy_s}{dx} = p_{s1}y_1 + p_{s2}y_2 + F_s(y_1, y_2, x) \qquad (s = 1, 2), \tag{1}$$

where p_{sj} are certain constants; F_s are holomorphic functions of all their arguments in a neighborhood of the point $x = y_1 = y_2 = 0$, vanishing together with their first partial derivatives with respect to y_1 and y_2 at this point.

We impose the following initial conditions:

$$y_s \to 0 \quad \text{as } x \to 0 \quad (s = 1, 2). \tag{2}$$

Here it is assumed that x tends to 0 in such a way that $\arg x$ remains bounded.

Picard already showed ([1]) that if the roots of the equation

$$\begin{vmatrix} p_{11} - \lambda & p_{12} \\ p_{21} & p_{22} - \lambda \end{vmatrix} = 0 \tag{3}$$

(which we shall denote by λ_1, λ_2) are not positive integers, then system (1) always has a unique holomorphic solution possessing property (2); if, however, the roots of equation (3) are simple and, while not being positive integers, do not admit any relation of the form

$$m + m_1\lambda_1 + m_2\lambda_2 = \lambda_j \qquad (j = 1, 2), \tag{4}$$

where m, m_1, m_2 are nonnegative integers whose sum is $\geqslant 2$, then our system has a certain family of solutions possessing property (2), which, depending on whether one or both roots have positive real parts, are represented in the form of series arranged in positive integral powers of the variables x, x^{λ_1} (or $x, x^{\lambda_1}, x^{\lambda_2}$), uniformly convergent for sufficiently small values of these variables and containing, respectively, one or two arbitrary constants. At the same time the abovementioned holomorphic solutions are obtained from these series if the arbitrary constants are set equal to zero.

The cases in which the roots of equation (3) are positive integers, or, without being such, admit relations of type (4), or, finally, turn out to be multiple (the so-called "doubtful" cases), Picard considers under the assumption that the functions F_s do not depend explicitly on x; in this case he shows that system (1) has a family of solutions which are, generally speaking, series in positive integral powers of the variables

$$x \quad \text{and} \quad z = x \ln x, \tag{5}$$

uniformly convergent for sufficiently small values of these variables and containing, as a rule, only one arbitrary constant, irrespective of whether one or both roots of equation (3) for system (1) have positive real parts. Moreover, Picard says nothing about the fact that, under certain conditions imposed on the functions F_s, the solutions constructed by him may contain holomorphic ones. The question of the existence, in the doubtful cases, of holomorphic solutions was partly studied by N. P. Erugin ([2]) and Bass ([3]).

2. In the present paper the following problems are posed:

1) To study the structure of the solutions of system (1) satisfying conditions (2) in a neighborhood of the point $x = 0$, considering all logically possible cases determined by the properties of the roots of equation (3).

2) To clarify the conditions under which holomorphic solutions exist in the doubtful cases.

3. The following theorem holds. If, in system (1), the functions $F_s(y_1, y_2, x)$ vanish together with their first partial derivatives with respect to y_1, y_2 at the point $x = y_1 = y_2 = 0$, then, under any assumptions concerning the roots of equation (3), this system has a family of solutions possessing the property

$$y_1 \to 0, \qquad y_2 \to 0,$$

when $x \to 0$ along some curve L on which the argument of x remains finite. This family contains one or two arbitrary constants, depending on whether one or both roots of equation (3) have positive real parts. Depending on the properties of these roots (which we shall denote by λ_1, λ_2), the constructed family of solutions can be represented by one of the following series, uniformly convergent for sufficiently small values of $|x|$ and for arbitrary values of the arbitrary constants. (For simplicity we assume that the linear part in system (1) has canonical form.)

1) $\operatorname{Re}\lambda_2 \leqslant 0 < \operatorname{Re}\lambda_1$; λ_1 is not a positive integer. Then the system

$$x\frac{dy_s}{dx} = \lambda_s y_s + F_s(y_1, y_2, x) \qquad (s = 1, 2) \tag{6}$$

has a solution of the form

$$y_s = \sum_{m+m_1=1}^{\infty} a_s^{(m,m_1)} x^m (x^{\lambda_1})^{m_1} \qquad (s = 1, 2),$$

where $a_1^{(0,1)} \equiv a$ is an arbitrary constant. If we put $a = 0$, we obtain the unique holomorphic solution.

2) $\operatorname{Re}\lambda_2 \leqslant 0 < \operatorname{Re}\lambda_1$; λ_1 is a positive integer. Then

$$y_s = \sum_{m+m_1=1}^{\infty} c_s^{(m,m_1)} x^m (x^{\lambda_1}\log x)^{m_1} \qquad (s = 1, 2),$$

where $c_1^{(\lambda_1,0)} \equiv c$ is an arbitrary constant. If the coefficient $c_1^{(0,1)}$ proves to be equal to zero, then these series determine a holomorphic solution containing the arbitrary constant c; otherwise no holomorphic solutions exist.

3) $0 < \operatorname{Re}\lambda_1 \leqslant \operatorname{Re}\lambda_2$; both roots are not positive integers and admit no relation of the form

$$m + m_1\lambda_1 = \lambda_2, \tag{7}$$

where m, m_1 are nonnegative numbers subject to the condition $m + m_1 \geqslant 2$.

Then

$$y_s = \sum_{m+m_1+m_2=1}^{\infty} a_s^{(m,m_1,m_2)} x^m (x^{\lambda_1})^{m_1} (x^{\lambda_2})^{m_2} \qquad (s = 1, 2),$$

where $a_1^{(0,1,0)} \equiv a_1$ and $a_2^{(0,0,1)} \equiv a_2$ are arbitrary constants. If these constants are set equal to zero, then we obtain the unique holomorphic solution.

4) $0 < \operatorname{Re}\lambda_1 \leqslant \operatorname{Re}\lambda_2$; λ_1 is a positive integer, while λ_2 is not. Then

$$y_s = \sum_{m+m_1+m_2=1}^{\infty} c_s^{(m,m_1,m_2)} x^m (x^{\lambda_1}\log x)^{m_1} (x^{\lambda_2})^{m_2} \qquad (s = 1, 2),$$

where $c_1^{(\lambda_1,0,0)} \equiv c_1$ and $c_2^{(0,0,1)} \equiv c_2$ are arbitrary constants. If the coefficient $c_1^{(0,1,0)}$ turns out to be equal to zero, then, putting $c_2 = 0$, we obtain a family of holomorphic solutions containing one arbitrary constant; if $c_1^{(0,1,0)} \neq 0$, then no holomorphic solutions exist.

5) $\lambda_1 \leqslant \lambda_2$ and both roots are positive integers. Then

$$y_s = \sum_{m+m_1+m_2=1}^{\infty} c_s^{(m,m_1,m_2)} x^m (x^{\lambda_1} \log x)^{m_1} \left[x^{\lambda_2} (\log x)^{\mu_1+1}\right]^{m_2},$$

where $c_1^{(\lambda_1,0,0)} \equiv c_1$ and $c_2^{(\lambda_2,0,0)} \equiv c_2$ are arbitrary constants; μ_1 is the integer part of the fraction λ_2/λ_1. If the coefficients $c_1^{(0,1,0)}$ and $c_2^{(0,0,1)}$ turn out to be equal to zero, then these series determine a holomorphic solution containing two arbitrary constants; if, however, at least one of the named coefficients is not equal to zero, then no holomorphic solutions exist.

6) $0 < \operatorname{Re} \lambda_1 \leqslant \operatorname{Re} \lambda_2$; λ_1 is not a positive integer, but the relations (7) are satisfied at least twice. Then the solution of system (6) can be represented by the same series as in case 5), with the same values of the arbitrary constants c_1, c_2; but here, unlike the preceding case, when λ_2 is not an integer there always exists a unique holomorphic solution, which is obtained from the indicated series if both constants are set equal to zero, while when λ_2 is an integer there either exists a family of holomorphic solutions with one arbitrary constant, or no such solutions exist.

7) $0 < \operatorname{Re} \lambda_1 \leqslant \operatorname{Re} \lambda_2$; λ_1 is not an integer, and relation (7) is satisfied only once. Then

$$y_s = \sum_{m+m_1+m_2=1}^{\infty} c_s^{(m,m_1,m_2)} x^m (x^{\lambda_1})^{m_1} (x^{\lambda_2} \log x)^{m_2} \qquad (s = 1, 2),$$

where $c_1^{(0,1,0)} \equiv c_1$ and $c_2^{(\lambda_2,0,0)} \equiv c_2$ are arbitrary constants. As regards the existence of holomorphic solutions, the remark made at the end of item 6 is valid.

8) Equation (3) has a multiple root with a simple elementary divisor; $\lambda_1 = \lambda_2 = \lambda$, $\operatorname{Re} \lambda > 0$. Then, for λ not an integer, the system

$$x \frac{dy_s}{dx} = \lambda y_s + F_s(y_1, y_2, x) \qquad (s = 1, 2)$$

has a solution of the form

$$y_s = \sum_{m+n=1}^{\infty} c_s^{(m,n)} x^m (x^{\lambda})^n \qquad (s = 1, 2),$$

where $c_1^{(0,1)} \equiv c_1$ and $c_2^{(0,1)} \equiv c_2$ are arbitrary constants, and, for integral λ, solutions of the form

$$y_s = \sum_{m+n=1}^{\infty} c_s^{(m,n)} x^m (x^\lambda \log x)^n;$$

the arbitrary constants will be $c_1^{(\lambda,0)} \equiv c_1$ and $c_2^{(\lambda,0)} \equiv c_2$. In the first case there always exists a unique holomorphic solution, which is obtained if c_1 and c_2 are set equal to zero; in the second case it exists only if both coefficients $c_1^{(0,1)}$ and $c_2^{(0,1)}$ turn out to be zero, and it contains two arbitrary constants.

9) Equation (3) has a multiple root with a non-simple elementary divisor; $\lambda_1 = \lambda_2 \equiv \lambda$, $\operatorname{Re}\lambda > 0$, λ nonintegral. Then the system

$$x\frac{dy_1}{dx} = \lambda y_1 + F_1(y_1, y_2, x),$$

$$x\frac{dy_2}{dx} = \lambda y_2 + y_1 + F_2(y_1, y_2, x)$$

has a solution of the form

$$y_s = \sum_{m+m_1+m_2=1}^{\infty} c_s^{(m,m_1,m_2)} x^m (x^\lambda)^{m_1} (x^\lambda \log x)^{m_2} \qquad (s = 1, 2),$$

where the arbitrary constants are

$$c_1^{(0,1,0)} \equiv c_1, \qquad c_1^{(0,0,1)} = 0, \qquad c_2^{(0,1,0)} \equiv c_2, \qquad c_2^{(0,0,1)} = c_1.$$

If c_1 and c_2 are set equal to zero, then we obtain the unique holomorphic solution.

10) Equation (3) has a multiple root λ with a non-simple elementary divisor, equal to an integral positive number. Then

$$y_s = \sum_{m+m_1+m_2=1}^{\infty} c_s^{(m,m_1,m_2)} x^m (x^\lambda \log x)^{m_1} (x^\lambda \log^2 x)^{m_2} \qquad (s = 1, 2);$$

the arbitrary constants will be $c_1^{(\lambda,0,0)} \equiv c_1$ and $c_2^{(\lambda,0,0)} \equiv c_2$. But if in this case there exists a holomorphic solution, then it contains only one arbitrary constant.

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CITED LITERATURE

[1] E. Picard, *Traité d' analyse*, 3, 1928. [2] N. P. Erugin, *Prikl. matem. i mekh.*, **16**, issue 4 (1952). [3] R. W. Bass, *Am. J. Math.*, **77**, No. 4 (1955).

Note: Figure translations are in progress. See original paper for figures.