THEORY OF ELASTICITY

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Abstract

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IMPACT ON AN INFINITE PLATE LYING ON AN ELASTIC LIQUID HALF-SPACE

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The action of a point impulse Q on an unbounded plate lying on a liquid half-space is considered. The latter is regarded as an elastic body for which the second Lamé constant $\mu = 0$. We use a cylindrical coordinate system with the z-axis directed into the depth of the half-space. The impulse applied to the plate will first be assumed distributed. Denoting the pressure on the plate due to the distributed impulse by $q(r,t)$, we set (1)

$$q(r,t) = \frac{Q}{2\pi}\,\frac{n^2}{(1-n^2r^2)^{3/2}}\,\delta^*(t), \tag{1}$$

where

$$\delta^*(t) = \begin{cases} \dfrac{1}{\varepsilon}, & 0 \le t \le \varepsilon, \\ 0, & t > \varepsilon. \end{cases} \tag{2}$$

Passing in (1) to the limit as $n \to \infty$ and $\varepsilon \to 0$, we obtain a concentrated impulse.

If the pressure on the plate from the side of the foundation is denoted by $p(r,t)$, then the equation of transverse vibrations of the plate will be

$$\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right]\right\} + \frac{\gamma h}{gD}\frac{\partial^2 w}{\partial t^2} = \frac{1}{D}q(r,t) - \frac{1}{D}p(r,t), \tag{3}$$

where $w(r,t)$ is the deflection; h is the thickness; γ is the specific weight of the material; $D = Eh^3/12(1-\nu^2)$ is the flexural rigidity of the plate.

The initial conditions for $w(r,t)$ are as follows:

$$w(r,0) = w_t(r,0) = 0. \tag{4}$$

The displacement components u_r and u_z in the half-space are determined as derivatives with respect to the corresponding coordinates of the displacement potential $\Phi(r,z,t)$, which satisfies the wave equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right)+\frac{\partial^2\Phi}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} \tag{5}$$

under the following initial and boundary conditions:

$$\Phi(r,z,0)=\Phi_t(r,z,0)=0; \tag{6}$$

$$\Phi_z(r,0,t)=w(r,t). \tag{7}$$

In equation (5), c is the propagation speed of the dilatational wave, with $c=\sqrt{\lambda/\rho}$, where λ is the Lamé constant and ρ is the density of the liquid.

The pressure on the plate from the side of the liquid is equal to:

$$p(r,t)=-\lambda\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right)+\frac{\partial^2\Phi}{\partial z^2}\right]_{z=0}=-\rho\left.\frac{\partial^2\Phi}{\partial t^2}\right|_{z=0}. \tag{8}$$

From the quantities $r, w, u_r, u_z, t, \Phi, n, \varepsilon$ we pass to dimensionless quantities

$$\bar{r}=\frac{r}{h},\qquad \bar{w}=\frac{w}{h},\qquad \bar{u}_r=\frac{u_r}{h},\qquad \bar{u}_z=\frac{u_z}{h},$$

$$\bar{t}=\frac{ct}{h},\qquad \bar{\Phi}=\frac{\Phi}{h^2},\qquad \bar{n}=hn,\qquad \bar{\varepsilon}=\frac{c\varepsilon}{h},$$

but in what follows we shall agree to omit the bars over dimensionless quantities.

From equations (3) and (5), taking (1) and (8) into account, on passing to dimensionless quantities we obtain

$$\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\right]\right\}+N\frac{\partial^2 w}{\partial t^2}-\frac{Qc}{2\pi D}\frac{n^2}{(1+n^2r^2)^{3/2}}\delta^*(t)-M\left.\frac{\partial^2\Phi}{\partial t^2}\right|_{z=0}=0; \tag{3'}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right)+\frac{\partial^2\Phi}{\partial z^2}=\frac{\partial^2\Phi}{\partial t^2}; \tag{5'}$$

$$N=\frac{\gamma c^2h^3}{gD}=12(1-\nu^2)\frac{\gamma c^2}{gE},\qquad M=\frac{\rho c^2h^3}{D}=12(1-\nu^2)\frac{\rho c^2}{E}. \tag{9}$$

We shall find solutions of equations (3′) and (5′) by using the method of incomplete separation of variables employed by G. I. Petrashen (^{1,2}). Put

$$w(r,t)=\int_0^\infty f(t,k)J_0(kr)\,dk,\qquad \Phi(r,z,t)=\int_0^\infty H(z,t,k)J_0(kr)\,dk. \tag{10}$$

Assume that differentiation under the integral signs in (10) is possible, which is legitimate, as the subsequent verification showed. From equations (3′) and (5′) we obtain a system of equations for determining $f(t,k)$ and $H(z,t,k)$:

$$k^4 f(t,k)+N\frac{\partial^2 f(t,k)}{\partial t^2}-\frac{Qc}{2\pi D}ke^{-k/n}\delta^*(t)-M\frac{\partial^2 H(z,t,k)}{\partial t^2}\bigg|_{z=0}=0;$$

$$-k^2H(z,t,k)+\frac{\partial^2 H(z,t,k)}{\partial z^2}=\frac{\partial^2 H(z,t,k)}{\partial t^2} \tag{11}$$

under the following initial and boundary conditions:

$$f(0,k)=f_t(0,k)=0,\qquad H(z,0,k)=H_t(z,0,k)=0; \tag{12}$$

$$H_z(0,t,k)=f(t,k). \tag{13}$$

Here the known relation (^{3}) has been used

$$\frac{1}{2\pi}\frac{n^2}{(1+n^2r^2)^{3/2}}=\frac{1}{2\pi}\int_0^\infty ke^{-k/n}J_0(kr)\,dk. \tag{14}$$

Put

$$\bar f(s,k)=\int_0^\infty f(t,k)e^{-st}dt,\qquad \bar H(z,s,k)=\int_0^\infty H(z,t,k)e^{-st}dt. \tag{15}$$

Applying the Laplace transform to equations (11), taking into account the initial conditions (12), we shall have:

$$k^4\bar f(s,k)+Ns^2\bar f(s,k)-\frac{Qc}{2\pi D}\frac{1}{\varepsilon s}(1-e^{-\varepsilon s})ke^{-k/n}-Ms^2\bar H(z,s,k)\big|_{z=0}=0;$$

$$\frac{\partial^2\bar H(z,s,k)}{\partial z^2}-(k^2+s^2)\bar H(z,s,k)=0. \tag{16}$$

The boundary condition (13) then takes the form:

$$\overline{H}_z(z,s,k)\Big|_{z=0} = \overline{f}(s,k). \tag{17}$$

The general solution of the second equation (16) will be

$$\overline{H}(z,s,k) = c_1(s,k)e^{-z\sqrt{k^2+s^2}} + c_2(s,k)e^{z\sqrt{k^2+s^2}}, \tag{18}$$

where the branch of the radical is fixed so that $\arg\sqrt{k^2+s^2} = 0$ for real positive s. The quantity $c_2(s,k)$ should be set equal to zero, since the entire half-space $z > 0$ is being considered. Determining $c_1(s,k)$ from condition (17), we obtain the expression $\overline{H}(z,s,k)$ in terms of $\overline{f}(s,k)$. Substituting the value $\overline{H}(0,s,k)$ into the first equation (16) and solving it with respect to $\overline{f}(s,k)$, we obtain

$$\overline{f}(s,k) = \frac{Qg}{2\pi\gamma c h^3}\, k e^{-kh}\, \frac{\sqrt{k^2+s^2}}{\left(\frac{1}{N}k^4+s^2\right)\sqrt{k^2+s^2}+\frac{M}{N}s^2}\,\frac{1}{\varepsilon s}\left(1-e^{-\varepsilon s}\right); \tag{19}$$

$$\overline{H}(z,s,k) = -\frac{Qg}{2\pi\gamma c h^3}\, k e^{-kh}\, \frac{e^{-z\sqrt{k^2+s^2}}}{\left(\frac{1}{N}k^4+s^2\right)\sqrt{k^2+s^2}+\frac{M}{N}s^2}\,\frac{1}{\varepsilon s}\left(1-e^{-\varepsilon s}\right). \tag{20}$$

By the inversion theorem,

$$f(t,k) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\overline{f}(s,k)e^{st}\,ds, \qquad H(z,t,k) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\overline{H}(z,s,k)e^{st}\,ds. \tag{21}$$

The functions $\overline{f}(s,k)$ and $\overline{H}(z,s,k)$ in the plane of the complex variable s have two branch points $s = \pm ik$ and poles at points that are zeros of the denominator. The zeros of the denominator are among the roots of the equation, cubic with respect to s^2:

$$\left(\frac{1}{N}k^4+s^2\right)^2(k^2+s^2) - \frac{M^2}{N^2}s^4 = 0, \tag{22}$$

which may be represented in the form

$$s^6 + a(k)s^4 + b(k)s^2 + d(k) = 0; \tag{22'}$$

$$a(k) = \frac{2}{N}k^4 + k^2 - \frac{M^2}{N^2}, \qquad b(k) = \frac{k^6}{N^2}(k^2 + 2N), \qquad d(k) = \frac{1}{N^2}k^{10}. \quad (23)$$

Depending on the values of k, equation (22) has, for s^2, either three real roots, one of which is always negative, or one negative real root and two complex conjugates. It is not difficult to verify that, of the real roots of equation (22), only the negative ones will be zeros of the denominator in $\overline{f}(s,k)$ and $\overline{H}(z,s,k)$. Moreover, in view of equation (22) one may conclude that if $s_1^2 < 0$ is its root, then necessarily $|s_1^2| < k^2$, i.e., the purely imaginary poles of $\overline{f}(s,k)$ and $\overline{H}(z,s,k)$ corresponding to this value s_1^2 lie between the branch points. Calculations have shown that if $0 < M/N \leq 1$, then $\overline{f}(s,k)$ and $\overline{H}(z,s,k)$ have, for all values of k, only two purely imaginary poles corresponding to the single negative root s_1^2 of equation (22). We denote the other two roots of this equation by s_2^2 and s_3^2.

The integrals in (23) are evaluated in the usual way, by introducing in the s-plane the corresponding contour, taking into account that

there are two branch points. We obtain:

$$f(t,k) = \frac{Qg}{2\pi\gamma ch^3} k e^{-k|n|} \left\{ R_f(t,k) + \right.$$

$$\left. + \frac{2}{\pi}\int_0^\infty F(k,m)\sqrt{m(m+2k)} \sin\left[(m+k)\left(t - \frac{\varepsilon}{2}\right)\right] \frac{\sin(m+k)\varepsilon/2}{(m+k)\varepsilon/2}\, dm \right\} \quad (24)$$

for $t - \varepsilon > 0$;

$$H(z,t,k) = -\frac{Qg}{2\pi\gamma ch^3} k e^{-k|n|} \left\{ R_H(z,t,k) + \right.$$

$$+ \frac{2}{\pi}\int_0^\infty \left[E(k,m)\sqrt{m(m+2k)} \cos z\sqrt{m(m+2k)} - \right.$$

$$\left. \left. - F(k,m) \sin z\sqrt{m(m+2k)} \right] \sin\left[(m+k)\left(t - \frac{\varepsilon}{2}\right)\right] \frac{\sin(m+k)\varepsilon/2}{(m+k)\varepsilon/2}\, dm \right\} \quad (25)$$

for $t - \varepsilon - z > 0$;

$$R_f(t,k) = \frac{1}{\sqrt{-s_1^2(s_1^2 - s_2^2)(s_1^2 - s_3^2)}} \frac{2\frac{M^2}{N^2}s_1^4}{\frac{1}{N}k^4 + s_1^2} \times$$

$$\times \sin\left[\left(t-\frac{\varepsilon}{2}\right)\sqrt{-s_1^2}\right]\frac{\sin\frac{\varepsilon}{2}\sqrt{-s_1^2}}{\frac{\varepsilon}{2}\sqrt{-s_1^2}}; \tag{26}$$

$$R_H(z,t,k)=$$

$$=\frac{-2\frac{M}{N}s_1^2}{\sqrt{-s_1^2(s_1^2-s_2^2)(s_1^2-s_3^2)}}e^{-z\sqrt{k^2+s_1^2}}\sin\left[\left(t-\frac{\varepsilon}{2}\right)\sqrt{-s_1^2}\right]\frac{\sin\frac{\varepsilon}{2}\sqrt{-s_1^2}}{\frac{\varepsilon}{2}\sqrt{-s_1^2}}; \tag{27}$$

$$E(k,m)=\frac{\frac{1}{N}k^4-(m+k)^2}{(m+k)^6-a(k)(m+k)^4+b(k)m(m+2k)+\frac{2}{N}k^8}; \tag{28}$$

$$F(k,m)=\frac{\frac{M}{N}(m+k)^2}{(m+k)^6-a(k)(m+k)^4+b(k)m(m+2k)+\frac{2}{N}k^8}. \tag{29}$$

Expressions for the deflection of the plate and for the displacement potential are obtained using relations (10).

If, in the formulas obtained, one passes to the limit as $n \to \infty$ and $\varepsilon \to 0$, then the solution for a concentrated impulse is obtained.

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Note: Figure translations are in progress. See original paper for figures.

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