M. Z. NARODETSKII

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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

THEORY OF ELASTICITY

M. Z. NARODETSKII

SOLUTION OF PROBLEMS OF THE PLANE THEORY OF ELASTICITY BY MEANS OF SPECIAL FUNCTIONS

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In the present article a method is indicated for the effective solution of a broad class of problems of the plane theory of elasticity for doubly connected regions (finite, infinite, and semi-infinite) bounded by circles and subjected to forces applied to the contours of the circles.

Consider an infinite plate S, weakened by two unequal holes. The origin and the direction of the coordinate axes of the plane $z = x + iy$, in which the plate is situated, as well as some notation, are indicated in Fig. 1.

Fig. 1. $z = x + iy$; $\varepsilon = \dfrac{R}{a+b}$; $c = \dfrac{r}{R}$

To determine the complex stress functions we have the boundary conditions (^{1})

$$\overline{\varphi(t)} + \bar{t}\,\varphi'(t) + \psi(t) = \overline{f_j(t)} + C_j \quad \text{on } L_j \ (j = 1, 2), \tag{1}$$

where $f_j(t)$ are prescribed functions of the complex coordinate t of the contour L_1 or L_2; C_j are certain constants to be determined.

Following D. I. Sherman (^{2}), we compose for $\varphi(z)$ the functional equations

$$\frac{1}{2\pi i}\int_L \frac{\overline{\varphi(t)} + \bar{t}\,\varphi'(t)}{t - z}\,dt = \sum_{m=1}^{2} \frac{1}{2\pi i}\int_{L_m} \frac{\overline{f_j(t)}}{t - z}\,dt - C_j \quad \text{in } \cdot L_j \ (j = 1, 2). \tag{2}$$

We introduce special functions α_n and β_n, regular, respectively, outside L_1 and L_2, and possessing certain convenient properties, which make it possible to reduce the problems to recurrence formulas

Fig. 2

Figure 2: Fig. 2

$$\alpha_n = \frac{\xi_{n-1}}{2\pi i} \int_{L_1} \frac{\overline{\beta_{n-1}(t)}}{t-z}\, dt, \qquad \beta_n = \frac{\xi^*_{n-1}}{2\pi i} \int_{L_2} \frac{\overline{\alpha_{n-1}(t)}}{t-z}\, dt \quad (n = 1, 2, \ldots),$$

$$\alpha_0 = -\frac{\varepsilon^{-1} c^{-1}}{2\pi i} \int_{L_1} \frac{r}{\overline{t}+b} \frac{dt}{t-z}, \qquad \beta_0 = \frac{\varepsilon^{-1}}{2\pi i} \int_{L_2} \frac{R}{\overline{t}-a} \frac{dt}{t-z}, \tag{3}$$

where ξ_n and ξ^*_n are certain constants.

The functions α_n and β_n, being of the same type in construction, rapidly tend to α and β—the limiting values of the corresponding functions with index n as $n \to \infty$, even when the boundaries are very close, and are representable, respectively, outside L_1 and L_2, in the form of series in powers of α and β.

Suppose that the load is continuous, and let us seek $\varphi(z)$ in the form

$$\varphi(z) = \chi(z) + a_{01}\alpha_0 + b_{01}\beta_0 + \sum_{k=1}^{s} \left(A_k \alpha^k + B_k \beta^k\right), \tag{4}$$

where a_{01}, b_{01}, A_k, and B_k are certain coefficients to be determined, and s is a certain fixed number.

We choose the function $\chi(z)$ in such a way (also using, for this purpose, the singularities of the functions α_0 and β_0 on L) that, after substituting $\varphi(z)$ into (2), the equations do not contain expressions of the form $(r/(z+b))^k$ for $j = 1$ or of the form $(R/(z-a))^k$ for $j = 2$, $k \neq 1$. Then, expanding the functions β_n and α_n contained in them in powers of β and α, and equating to zero in the first equation the expressions at β^k and $r(z+b)$, and in the second those at α^k and $R/(z-a)$, we obtain recurrence formulas for the indicated coefficients. Comparing the free terms in the same equations, we find the constants C_1 and C_2.

Fig. 2

For loads distributed over separate portions and for concentrated loads, the solution is not essentially changed. In this case it is necessary in (1) to introduce the substitution $\varphi(z) = \varphi_1(z) + \varphi^*(z)$ and $\psi(z) = \psi_1(z) + \psi^*(z)$, where $\varphi^*(z)$ and $\psi^*(z)$ are the zeroth or the sum of the zeroth and first Schwarz approximations, respectively, for the cases when loads of this kind are applied to one or to both contours.

Let $r = R$, and at the point $t = a - R$ of the contour L_1 let a concentrated compressive force P be applied (Fig. 2). Taking into account the zeroth Schwarz approximation, instead of (1) we obtain

$$\overline{\varphi_1(t)} + \bar{t}\,\varphi_1'(t) + \psi_1(t) = \overline{f_j^*(t)} + C_j^* \quad \text{on } L_j \ (j = 1, 2), \tag{5}$$

where $f_1^*(t) = 0$ and

$$\overline{f_2^*(t)} = \frac{P}{2\pi}\left(-\ln\frac{a_0' - (t+a)}{t+a}\,\frac{2a-R}{t-a+R} + \frac{R}{t-a} + \frac{2a-R-a_0'}{t-a+R} + \mu_0'\frac{R}{t+a}\right), \quad a_0' = R\mu_0', \quad \mu_0' = \frac{\varepsilon}{1-\varepsilon}. \tag{6}$$

The functions α_n and β_n $(n = 0, 1, \ldots)$ have the form

$$\alpha_n = \frac{a_n}{a_n + (z-a)}, \qquad \beta_n = \frac{a_n}{a_n - (z+a)}, \qquad a_n = R\mu_n,$$

$$\mu_n = \frac{\varepsilon}{1 - \varepsilon\mu_{n-1}}, \qquad \mu_0 = \varepsilon, \qquad \xi_n = \frac{2a - a_n}{a_n}, \tag{7}$$

$$\alpha = \frac{c}{c + (z-a)}, \qquad \beta = \frac{c}{c - (z+a)}, \qquad c = R\mu, \qquad \mu = \frac{\varepsilon}{1 - \varepsilon\mu} < 1, \qquad \xi = \frac{2a-c}{c}.$$

Put in (3) $\alpha_0^* = 0$ and

$$\beta_0^* = \frac{\mu_0'^{-1}}{2\pi i}\int_{L_2}\frac{R}{\bar{t} - a + R}\,\frac{dt}{t-z}. \tag{8}$$

The expressions for α_n and β_n obtained from the remaining formulas (3) coincide in form with α_n and β_n, and their limiting values are also equal, respectively, to α and β. We shall agree to attach the sign $'$ to constants pertaining to the functions α_n^* and β_n.

Further, we have

$$\alpha_n^k = \lambda_n^k \sum_{k_1=k}^{\infty} C_{-k}^{k_1-k}\nu_n^{k_1-k}\alpha^{k_1}, \quad \ln\frac{a_n' + (z-a)}{a_m + (z-a)} = -\sum_{k=1}^{\infty}\frac{1}{k}\left(\nu_{n'}^k - \nu_m^k\right)\alpha^k \quad \text{outside } L_1;$$

$$\beta_n^k = \lambda_n^k \sum_{k_1=k}^{\infty} C_{-k}^{k_1-k}\nu_n^{k_1-k}\beta^{k_1}, \quad \ln\frac{a_n' - (z+a)}{a_m - (z+a)} = -\sum_{k=1}^{\infty}\frac{1}{k}\left(\nu_{n'}^k - \nu_m^k\right)\beta^k \quad \text{outside } L_2; \tag{9}$$

$$\lambda_n = \mu_n/\mu, \qquad \nu_n = 1 - \lambda_n \qquad (k = 1, 2, \dots;\ m, n = 0, 1, \dots).$$

Replacing in the first equalities (9) μ_n, λ_n, and ν_n, respectively, by μ'_n, λ'_n, and ν'_n, we obtain analogous expansions for α^*_n and β^*_n.

We shall seek the function $\varphi_1(z)$ in the form

$$\varphi_1(z) = \frac{P}{2\pi}\left(a_{01}\alpha_0 + b_{01}\beta_0 + \ln\frac{a'_1 + (z-a)}{a_0 + (z-a)}\frac{a'_0 - (z+a)}{z+a} - \right.$$

$$\left. - \varepsilon^2\alpha_0^2 + (1 - \mu_0'^2)\beta^*_0 + \sum_{k=1}^{s}\left(A_k\alpha^k + B_k\beta^k\right)\right). \qquad (10)$$

The functional equations (2), after substituting into them (6), (9), and (10), and comparing in both sides of these equations the expressions in β and α in equal powers, respectively for $j = 1$ and $j = 2$, lead to recurrence formulas for the coefficients A_k and B_k. These formulas, with the aid of the notation

$$a'_{01} = a_{01} + b_{01}, \qquad a''_{01} = a_{01} - b_{01},$$

$$A'_k = A_k + B_k, \qquad A''_k = A_k - B_k \qquad (k = 1, 2, \dots, s) \qquad (11)$$

are conveniently written in the form

$$\sum_{k_1=k}^{s} A'_{k_1}\left(C^k_{k_1}\left(1 - (-1)^{k_1+k}\xi^{-(k_1+k)}\right) - k_1\left(\xi^{-k+1} - (-1)^{k_1+k}\xi^{-(k_1+1)}\right)\right) =$$

$$= \xi^{-k}\left(\delta^+_k + \Delta^+_k\right); \qquad (12)$$

$$\sum_{k_1=k}^{s} A''_{k_1}\left(C^k_{k_1}\left(1 - (-1)^{k_1+k}\xi^{-(k_1+k)}\right) + k_1\left(\xi^{-k+1} - (-1)^{k_1+k}\xi^{-(k_1+1)}\right)\right) =$$

$$= \xi^{-k}\left(\delta^-_k + \Delta^-_k\right), \qquad (13)$$

where the notation has been introduced

$$\delta^+_k = a'_{01}\left(\lambda_0\left(\varepsilon^{-2}(1 - \xi_0^{-2}) + \lambda_0(1 - \xi_0^{-1})\right)\nu_0^{-1}C^{k-2}_{-2}\nu_0^{k-1} + \xi_0^{-1}\lambda_1\nu_1^{k-1}\right); \qquad (14)$$

$$\delta_k^- = -a_{01}'' \left(\lambda_0 \left(\varepsilon^{-2}(1-\xi_0^{-2}) + \lambda_0(1-\xi_0^{-1})\right) \nu_0^{-1} C_{-2}^{k-2} \nu_0^{k-1} - \xi_0^{-1}\lambda_1 \nu_1^{k-1}\right); \quad (15)$$

$$\Delta_k^+ = \lambda_0' \left(-1 + \mu_0'^{-1}\mu_2'^{-1} - \mu_1'^2 + \xi_0'^{-2} - \right.$$

$$-(\mu_0'^{-1} - \mu_0')(1-\mu_1')\lambda_0'\nu_0'^{-1}C_{-2}^{k-2}\nu_0'^{k-1} - \frac{1}{k}\left(\nu_2'^k - \nu_1^k\right) +$$

$$\begin{gathered}
+\varepsilon^2\xi_0^{-2}(2-\lambda_1\nu_1^{-1}C_{-2}^{k-2})\lambda_1\nu_1^{k-1} + \{\lambda_0(1-\xi_0^{-1} + 2\xi_0^{-3} + 2(1-\xi_0^{-2})\lambda_0\nu_0^{-1}C_{-2}^{k-2} \\
+2\xi_0^{-1}\lambda_0^2\nu_0^{-2}C_{-3}^{k-3})\nu_0^{k-1} - \lambda_1'(\mu_1'^{-1}(\mu_0' - \mu_2') \\
+(1-\mu_0'^2)\xi_0'^{-1})\nu_1'^{k-1} + \frac{1}{k}(\nu_1'^k - \nu_0'^k)\}. \qquad (16)
\end{gathered}$$

Comparing in the same equations the expressions for $R/(z+a)$ and $R/(z-a)$, respectively for $j=1$ and $j=2$, as well as the free terms, we obtain

$$\begin{gathered}
\varepsilon^{-1}a_{01}'(2-\xi_0^{-2}) + \mu^{-1}\sum_{k=1}^{s} kA_k'(1+(-1)^k\xi^{-(k+1)}) = \\
= \varepsilon(4+2\xi_0^{-3}) - \mu_1 + \mu_1' + \mu_2' - (\mu_0'^{-1} - \mu_0')(1-\xi_0'^{-2}); \qquad (17)
\end{gathered}$$

$$\begin{gathered}
\varepsilon^{-1}\xi_0^{-2}a_{01}'' + \mu^{-1}\sum_{k=1}^{s} kA_k''(1-(-1)^k\xi^{-(k+1)}) = \\
= 1 - 2\varepsilon(1+\xi_0^{-3}) - 3\mu_0' + \mu_1 + \mu_1'^{-1} + \mu_3'^{-1} + (\mu_0'^{-1} - \mu_0')\xi_0'^{-2}; \qquad (18)
\end{gathered}$$

$$\begin{gathered}
C_1' = C_1^* + C_2^* = a_{01}'(1-\xi_0^{-1}) + \sum_{k=1}^{s} A_k'(1+(-1)^k\xi^{-k}) - \\
-\varepsilon^2(1+\xi_0^{-2}) + (1-\mu_0'^2)(1-\xi_0'^{-1}) + \ln\frac{\mu_1}{\mu_2'}; \qquad (19)
\end{gathered}$$

$$\begin{gathered}
C_1'' = C_1^* - C_2^* = a_{01}''(1+\xi_0^{-1}) + \sum_{k=1}^{s} A_k''(1-(-1)^k\xi^{-k}) - \\
-\varepsilon^2(1-\xi_0^{-2}) - (1-\mu_0'^2)(1+\xi_0'^{-1}) - \ln\frac{\mu_1}{\mu_2'}. \qquad (20)
\end{gathered}$$

The coefficients A_k' $(k = s,\ s{-}1, \dots, 2)$ and A_k'' $(k = s,\ s{-}1, \dots, 1)$ are determined from the recurrence formulas (12) and (13), where the former contain the as yet unknown coefficient a'_{01}, and the latter a''_{01}. The coefficient a'_{01} is found from formula (12) for $k = 1$, and a''_{01} from (18). After this, from formulas (17), (19), and (20), we successively find the coefficient A'_1 and the constants C'_1 and C''_1. Finally, returning to formulas (11), (19), and (20), we find the coefficients A_k and B_k $(k = 1, 2, \dots, s)$, a_{01} and b_{01}, as well as the constants C^*_1 and C^*_2.

Figure 2 gives the diagram of σ_θ (θ is the polar angle, measured from the polar axis r' with origin at the point $z = -a$) at various points of L_2 for $R/a = 0.9$ and $s = 11$. We note that the boundary condition (5) on L_1 is satisfied here exactly for any fixed s, and on L_2, for the chosen s, the greatest deviation from the boundary condition does not exceed 0.002%.

The numerical results obtained can be extended to a plate of finite width if the width of the plate considerably exceeds the diameter of the holes.

Unlike the known methods for solving analogous problems, the present method leads to rapidly convergent processes for any closeness of the boundaries and for arbitrary loads.

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Note: Figure translations are in progress. See original paper for figures.

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