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Abstract

Full Text

MATHEMATICS

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ON DERIVED OPERATIONS AND PREDICATES

Let A be an algebra with fundamental operations $f_i(x_1, \dots, x_{m_i})$, $i = 1, 2, \dots$. The principal type of derived operations defined on A is terms, or polynomials. But the operations f_i may also be regarded as predicates $P_i(x_1, \dots, x_{m_i}, y)$ defined on A and meaning that $f_i(x_1, \dots, x_{m_i}) = y$. Every well-formed formula $\mathfrak{A}(x_1, \dots, x_n)$ of the restricted predicate calculus (RPC), composed of predicate symbols P_i and containing free individual variables x_1, \dots, x_n , may be regarded as a derived predicate on A . It may happen that \mathfrak{A} represents on A an operation in the sense indicated above. Thus, in addition to the formation of terms, there arises another way of obtaining new operations on A .

In no. 1 we establish the general form of operations obtained by means of RPC formulas in a class of algebras characterized by universal axioms. In no. 2 an abstract characterization is given of predicates representable by conjunctions of universal RPC formulas, and in no. 3 the results obtained are applied to finding the general form of derived operations satisfying certain additional requirements.

No. 1. Let, for an arbitrary RPC formula \mathfrak{A} with free variables x_1, \dots, x_m, y , the symbol $\Phi(\mathfrak{A})$ denote the expression

$$(x_1) \cdots (x_m)(u)(v)(\exists y)[\mathfrak{A}(x_1, \dots, x_m, y) \& \&(\mathfrak{A}(x_1, \dots, x_m, u) \& \mathfrak{A}(x_1, \dots, x_m, v) \rightarrow u = v)].$$

Formulas of the form $(y_1) \cdots (y_p)\mathfrak{B}(x_1, \dots, x_n, y_1, \dots, y_p)$, where \mathfrak{B} contains no quantifiers, will be called **universal**, and formulas containing no free variables will be called **axioms**.

Theorem 1. Let \mathfrak{K} be a conjunction of a finite or infinite number of universal axioms, and let \mathfrak{A} be an RPC formula with free individual variables x_1, \dots, x_m, y , where the axioms from \mathfrak{K} and \mathfrak{A} contain only the predicate symbols P_1, \dots, P_s . If the expression

$$\Phi(P_1) \& \dots \& \Phi(P_s) \& \mathfrak{K} \rightarrow \Phi(\mathfrak{A}) \tag{1}$$

is a provable RPC formula, then there exist formulas $\mathfrak{A}_i(x_1, \dots, x_m)$ and terms $g_i(x_1, \dots, x_m)$, $j = 1, \dots, t$, composed of the operation symbols corresponding to the predicates P_i , such that the formulas

$$\mathfrak{A}_1(x_1, \dots, x_m) \vee \dots \vee \mathfrak{A}_t(x_1, \dots, x_m), \quad (2)$$

$$\mathfrak{A}_i(x_1, \dots, x_m) \& \mathfrak{A}_j(x_1, \dots, x_m) \sim g_i(x_1, \dots, x_m) = g_j(x_1, \dots, x_m), \quad (3)$$

$$\mathfrak{A}(x_1, \dots, x_m, y) \sim \quad (4)$$

$$\sim [(\mathfrak{A}_1(x_1, \dots, x_m) \sim y = g_1(x_1, \dots, x_m)) \& \dots \& (\mathfrak{A}_t \sim y = g_t)],$$

will be consequences of the axioms $\mathfrak{S} = \{\Phi(P_1), \dots, \Phi(P_s), \mathfrak{K}\}$.

For the proof, denote by h_α all possible terms in the variables x_1, \dots, x_m , and consider the system of sentences

$$\mathfrak{S}_1 = \{\sim \mathfrak{A}(x_1, \dots, x_m, h_\alpha)\}, \mathfrak{S}.$$

Suppose that this system has some model A . In view of the axioms $\Phi(P_1), \dots, \Phi(P_s)$, the model A will be an algebra with operations f_1, \dots, f_s corresponding to the predicates P_1, \dots, P_s . The subalgebra B generated in A by the elements x_1, \dots, x_m will, by virtue of the universality of the axioms of the system \mathfrak{K} , also be a model for \mathfrak{S}_1 . Since B satisfies the premise of expression (1), in B there will be an element $y = h_\lambda$ for which $\mathfrak{A}(x_1, \dots, x_m, h_\lambda)$ is true, contrary to the assumption. From the inconsistency of \mathfrak{S}_1 follows the inconsistency of a suitable finite part of it, i.e. from the axioms \mathfrak{S} , for suitable terms $g_1 = h_{\alpha_1}, \dots, g_t = h_{\alpha_t}$, the formula follows

$$\mathfrak{A}(x_1, \dots, x_m, y) \rightarrow y = g_1 \vee \dots \vee y = g_t. \quad (5)$$

Putting now

$$\mathfrak{A}_i(x_1, \dots, x_m)' = (z)(\mathfrak{A}(x_1, \dots, x_m, z) \sim y = g_i(x_1, \dots, x_m)) \quad (i = 1, \dots, t),$$

we obtain (2), (3), (4). Indeed, let x_1, \dots, x_m be arbitrary elements of an algebra satisfying the axioms \mathfrak{K} . By the condition there will be an element y for which $\mathfrak{A}(x_1, \dots, x_m, y)$ is true. From (5) it follows that $y = g_\lambda(x_1, \dots, x_m)$ for some λ . Therefore $\mathfrak{A}_\lambda(x_1, \dots, x_m)$ is true, and (2) is proved. Suppose that for some x_1, \dots, x_m , y the expression $\mathfrak{A}(x_1, \dots, x_m, y)$ is false. Then there will be z, λ for which $\mathfrak{A}(x_1, \dots, x_m, z)$ is true, $z = g_\lambda(x_1, \dots, x_m)$, $z \neq y$, and therefore $\mathfrak{A}_\lambda(x_1, \dots, x_m)$ is also true. Then the expression $\mathfrak{A}_\lambda(x_1, \dots, x_m) \sim y = g_\lambda(x_1, \dots, x_m)$ is false. Together with it the entire right-hand side of the equivalence (4) is false. Thus, from the falsity of the left-hand side of formula (4)

follows the falsity of its right-hand side. The converse assertion and formula 3 are proved similarly.

No. 2. Let K be some class of abstract models of the form $\langle M; P_1, \dots, P_s \rangle$. Suppose that on each model of the class K there is given, in some way, not necessarily by a formula of the restricted predicate calculus and not necessarily in a unique way, one more predicate $P(x_1, \dots, x_n)$. As an example one may take the predicate “ x is incomparably smaller than y ,” which has meaning in every ordered group, although it cannot be expressed by a formula of the restricted predicate calculus. We shall call P **invariant with respect to passing to K -submodels** if from the truth of $P(x_1, \dots, x_n)$ in a K -model containing x_1, \dots, x_n follows the truth of $P(x_1, \dots, x_n)$ in every one of its K -submodels containing x_1, \dots, x_n . Similarly, we shall call a predicate P **invariant with respect to passing to K -extensions** if from the truth of $P(x_1, \dots, x_n)$ in some K -model M follows the truth of $P(x_1, \dots, x_n)$ in every K -model containing M as its submodel. We shall call a predicate P **formal** if there exists a formula of the restricted predicate calculus $\mathfrak{A}(x_1, \dots, x_n)$ for which $P(x_1, \dots, x_n) \sim \mathfrak{A}(x_1, \dots, x_n)$ on every K -model. Obviously, the invariance of P with respect to passing to submodels is equivalent to the invariance of $\sim P$ with respect to passing to extensions. A formal predicate invariant with respect to passing to extensions is called, by A. Robinson (3), **stable on K** .

A predicate $P(x_1, \dots, x_n)$, the truth or falsity of which is not necessarily defined for all x_1, \dots, x_n from some set M , will be called a **partial predicate** on M . A mapping σ of a set M with a partial predicate P into a set N with a predicate of the same name is called a P -homomorphism if, from the truth of $P(x_1, \dots, x_n)$ in M , there follows the definedness and truth of $P(x_1^\sigma, \dots, x_n^\sigma)$. Let us agree to say that the set M is P -embedded in N by the mapping σ , if σ maps M onto M^σ one-to-one and if from the definedness

of the definedness of $P(x_1, \dots, x_n)$ on M implies the definedness of $P(x_1^\sigma, \dots, x_n^\sigma)$ on N and its equivalence with $P(x_1, \dots, x_n)$.

Theorem 2. In order that, on the class K^* of models with basic predicates P_1, \dots, P_s, P , there hold a formula

$$P(x_1, \dots, x_n) \sim \&(x_{n+1}) \dots (x_{p_\alpha}) \mathfrak{B}_\alpha(x_1, \dots, x_n, x_{n+1}, \dots, x_{p_\alpha}), \quad (6)$$

in which the conjunction may be infinite, and the \mathfrak{B}_α are open formulas of the first-order predicate calculus not containing the sign P , it is necessary and sufficient that the class K^* have the following properties: if σ is a (P_1, \dots, P_s) -isomorphism of some (P_1, \dots, P_s) -model M , equipped with an additional partial predicate P , onto a suitable K^* -model M^* , and at the same time every finite (P_1, \dots, P_s, P) -submodel of the partial model M is embeddable in some K^* -model, then σ is a P -homomorphism of M onto M^* .

Necessity. Suppose that on M^* formula (6) holds, and for some a_1, \dots, a_n from M the expression $P(a_1, \dots, a_n)$ is defined, while $P(a_1^\sigma, \dots, a_n^\sigma)$ is false. Since σ is

a (P_1, \dots, P_s) -isomorphism, the formulas \mathfrak{B}_x do not contain the symbol P ; hence on M the expression occurring on the right-hand side of the equivalence (6) is false. Therefore there will be a λ and such a_{n+1}, \dots, a_p in M that $\mathfrak{B}_\lambda(a_1, \dots, a_p)$ is false. By the hypothesis, the submodel M_λ , consisting in M of the elements a_1, \dots, a_p , is embedded in some K^* -model N , in which for P the expression (6) holds. Since $\mathfrak{B}_\lambda(a_1, \dots, a_p)$ is false, $P(a_1, \dots, a_n)$ is also false in N , and therefore also false in M .

Sufficiency. Denote by $\mathfrak{A}_\lambda(x_1, \dots, x_n)$ all possible universal formulas not containing the symbol P , for which in K^* the relation

$$(x_1) \dots (x_n)[P(x_1, \dots, x_n) \rightarrow \mathfrak{A}_\lambda(x_1, \dots, x_n)]. \quad (7)$$

is satisfied. Suppose that there exists a K^* -model M^* in which the system $\{\mathfrak{A}_\lambda(a_1, \dots, a_n), \sim P(a_1, \dots, a_n)\}$ is satisfied, where a_1, \dots, a_n are individual constants. Let M be the same model M^* , but with the predicate P changed: in M put $P(a_1, \dots, a_n)$ true, and for all other values of the arguments regard P as undefined. The identity mapping σ of M onto M^* is a (P_1, \dots, P_s) -isomorphism. If every finite submodel of the model M were embeddable in a suitable K^* -model, then by hypothesis the mapping σ would be a homomorphism, which it is not, since in M the expression $P(a_1, \dots, a_n)$ is true, while in M^* it is false. Therefore there is a finite submodel M_0 of the model M not embeddable in any K^* -model. Let the diagram ⁽²⁾ of the model M_0 be $P(a_1, \dots, a_n) \& R(a_1, \dots, a_n, \dots, a_q)$, where R is an open formula not containing the symbol P . In K^* we have $P(a_1, \dots, a_n) \rightarrow (a_{n+1}) \dots (a_q)R$, i.e., for some λ we have

$$\mathfrak{A}_\lambda(x_1, \dots, x_n) = (a_{n+1}) \dots (a_q)\overline{R}(x_1, \dots, x_n, a_{n+1}, \dots, a_q). \quad (8)$$

But in M^* we have simultaneously $\mathfrak{A}_\lambda(a_1, \dots, a_n)$ and $R(a_1, \dots, a_n, a_{n+1}, \dots, a_q)$, which contradicts (8). This shows that the model M^* does not exist, i.e. in K^* , from $\&\mathfrak{A}_\lambda(a_1, \dots, a_n)$ there follows $P(a_1, \dots, a_n)$, which together with (7) gives (6).

Taking in Theorem 2 as P a 0-place predicate and denoting by L the collection of those K^* -models on which the predicate P is true, we arrive at theorem A. Tarski's ⁽⁴⁾: a subclass L of models of a class K is then and only then singled out from K by a system of universal axioms when, from the embeddability of every finite submodel of an arbitrary K -model M in a suitable L -model, it follows that M is an L -model.

Assuming in Theorem 2 the predicate P to be formula-definable, we obtain as a consequence A. Robinson's theorem ⁽³⁾: in order that a formula-definable predicate \mathfrak{A} on an axiomatizable class of models K be equivalent to a universal formula-definable predicate, it is necessary and sufficient that—

the predicate \mathfrak{A} would be invariant with respect to passage to K -submodels.

No. 3. Open formulas are the simplest formal predicates invariant with respect to passage to sub- and supermodels in arbitrary classes K . If the class K is characterized by universal axioms, then open formulas will be the only formal predicates possessing the indicated properties. However, in the class of algebras this is no longer true, since all terms possess the required invariance.

Theorem 3. *If, in a universally axiomatizable class of algebras K , a formula $\mathfrak{A}(x_1, \dots, x_n)$ is a predicate invariant with respect to passage to super- and subalgebras, then \mathfrak{A} is equivalent in K to an open formula composed of expressions of the form $g = h$, where g, h are certain terms.*

By the second corollary of Theorem 2, for \mathfrak{A} in the class K we have the representation

$\mathfrak{A}(x_1, \dots, x_n) \sim (y_1) \dots (y_p) \mathfrak{B}(x_1, \dots, x_n, y_1, \dots, y_p)$. Let a_1, \dots, a_n be elements of an arbitrary K -algebra A , and let B be the subalgebra generated in A by the elements a_1, \dots, a_n . By assumption, the value of $\mathfrak{A}(a_1, \dots, a_n)$ in A coincides with the value of this expression in B , i.e. in K we have

$$\& \mathfrak{B}(a_1, \dots, a_n, g_1^{\nu}, \dots, g_p^{\nu}) \rightarrow \mathfrak{A}(a_1, \dots, a_n), \quad (9)$$

where the conjunction is taken over all possible sets of terms $g_1^{\nu}, \dots, g_p^{\nu}$. Since formula (9) must follow from the system of axioms defining K , it must already hold for the conjunction of a finite number of terms, as was required.

A predicate P , defined on the class of models K , will be called **multiplicatively invariant** if

$P(\langle a_1, b_1 \rangle, \dots, \langle a_n, b_n \rangle)$ is equivalent to $P(a_1, \dots, a_n) \& P(b_1, \dots, b_n)$, where $\langle a_i, b_i \rangle$ are elements of the direct product of K -models A, B ($a_i \in A, b_i \in B$), under the condition that $A \times B \in K$.

Theorem 4. *In a quasiprimitive class ⁽¹⁾ of algebras K , a formal predicate $\mathfrak{A}(x_1, \dots, x_m, y)$ is a multiplicatively invariant operation if and only if \mathfrak{A} is representable by a term.*

According to Theorem 1, for \mathfrak{A} formula (5) holds. If each of the formulas $\mathfrak{A}(x_1, \dots, x_m, y) \rightarrow y = g_i(x_1, \dots, x_m)$ is not satisfied on K , then there exist K -algebras A_i , containing a_1^i, \dots, a_m^i, b^i , in which

$\mathfrak{A}(a_1^i, \dots, a_m^i, b^i)$ is true and $b^i \neq g_i(a_1^i, \dots, a_m^i)$. Putting $a_\alpha = \langle a_\alpha^1, \dots, a_\alpha^t \rangle$, in $A_1 \times \dots \times A_t$ we obtain $\mathfrak{A}(a_1, \dots, a_m, b)$ and $b \neq g_i(a_1, \dots, a_m)$, $i = 1, \dots, t$, in contradiction with (5).

Theorem 5. *In order that a derived operation represented on a quasiprimitive class of algebras K by a formula $\mathfrak{A}(x_1, \dots, x_m, y)$ be a term, it is necessary and sufficient that it be homomorphically invariant, i.e. that for every homomorphism σ of a K -algebra A onto an arbitrary K -algebra B , from the truth of $\mathfrak{A}(a_1, \dots, a_m, a)$ in A there follows the truth of $\mathfrak{A}(a_1^\sigma, \dots, a_m^\sigma, a^\sigma)$ in B .*

Consider a K -free algebra A with free generators a_1, \dots, a_m, \dots ⁽¹⁾. By assumption there is a term $g(a_1, \dots, a_m)$ for which $\mathfrak{A}(a_1, \dots, a_m, g)$ is true. Since there exists a homomorphism of A into any K -algebra B , in which the a_i^g may be chosen arbitrarily, it follows that in any K -algebra $\mathfrak{A}(a_1, \dots, a_m, g)$ is true, i.e. in K we have

$$\mathfrak{A}(a_1, \dots, a_m, a) \sim a = g(a_1, \dots, a_m).$$

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- ¹ A. I. Mal' tsev, DAN, 108, No. 2, 187 (1956).
- ² A. Robinson, Indag. Math., 18, No. 1, 47 (1956).
- ³ A. Robinson, J. Symb. Logic, 21, No. 1, 33 (1956).
- ⁴ A. Tarski, Proc. Acad. v. Wetensch., 57, No. 5, 582 (1954).

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