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Abstract

Full Text

MATHEMATICS

G. V. KORITSKII

ON THE CURVATURE OF LEVEL LINES UNDER UNIVALENT CONFORMAL MAPPINGS

(Presented by Academician M. A. Lavrent' ev, 12 III 1957)

The theorems set forth below are a continuation of results obtained earlier by the author ([1]).

We study the curvature K_ρ of level lines (images of the circles $|\zeta| = \rho =$ const, $\zeta = \rho e^{i\varphi}$) in the class Σ of functions

$$F(\zeta) = \zeta + a_0 + \sum_{n=1}^{\infty} \frac{a_n}{\zeta^n},$$

univalent and regular in the domain $|\zeta| = \rho > 1$, except for the simple pole $\zeta = \infty$, and also in the subclass Σ_2 of the class Σ, consisting of functions

$$F_2(\zeta) = \zeta + \sum_{n=1}^{\infty} \frac{a_n}{\zeta^{2n-1}},$$

and in the subclasses Σ_p^*, $p = 1, 2, \dots$, consisting of functions

$$w = F_p^*(\zeta) = \zeta + \sum_{n=1}^{\infty} \frac{a_n}{\zeta^{np-1}},$$

which map the domain $\rho > 1$ onto domains with p-fold rotational symmetry and with complements star-shaped with respect to the point $w = 0$.

The results obtained constitute the following theorems.

Theorem 1. In the class Σ the following sharp estimate holds:

$$K_\rho \leqslant \frac{\rho(\rho^2+1)}{(\rho^2-1)^2}. \tag{I}$$

Proof. It is known that

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$$K_\rho = R\left\{1+\frac{\zeta F''(\zeta)}{F'(\zeta)}\right\}\frac{1}{|\zeta|\,|F'(\zeta)|}. \tag{1}$$

From the estimate of G. M. Goluzin ([2])

$$\left|\frac{\zeta F''(\zeta)}{F'(\zeta)}+\frac{4\rho^2-2}{\rho^2-1}-\frac{4\rho^2}{\rho^2-1}\frac{E(1/\rho)}{K(1/\rho)}\right| \leqslant \frac{4\rho^2}{\rho^2-1}\left\{1-\frac{E(1/\rho)}{K(1/\rho)}\right\},$$

where

$$E\left(\frac{1}{\rho}\right)=\int_0^1\sqrt{\frac{1-x^2/\rho^2}{1-x^2}}\,dx, \qquad K\left(\frac{1}{\rho}\right)=\int_0^1\frac{dx}{\sqrt{(1-x^2)(1-x^2/\rho^2)}},$$

it follows that

$$R\left\{\frac{\zeta F''(\zeta)}{F'(\zeta)}\right\}\leqslant\frac{2}{\rho^2-1}. \tag{2}$$

In addition, we have the well-known Löwner estimate ([3])

$$1-\frac{1}{\rho^2}\leqslant|F'(\zeta)|. \tag{3}$$

From (1), (2), and (3) we obtain (I).

The right-hand side of (I) is attained by the function $F(\zeta)=\zeta+\alpha_0+1/\zeta$ at the point $\zeta=\rho$.

Corollary. Since the function $F_2(\zeta)=\zeta+1/\zeta$ is extremal in Theorem 1, the theorem obviously remains valid for the subclasses $\Sigma_2,\ \Sigma_1^*,\ \Sigma_2^*$, to which this function belongs.

Theorem 2. In the subclasses $\Sigma_p^*,\ p=2,3,\dots$, the sharp estimate holds

$$K_\rho\leq\frac{\rho\left[\rho^{2p}+2(p-1)\rho^p+1\right]}{(\rho^p-1)^2(\rho^p+1)^{2/p}}. \tag{II}$$

Proof. From the known relation $F_p^*(\zeta)=\sqrt[p]{F_1^*(\zeta^p)}$ we obtain

$$R\left\{1+\frac{\zeta F_p^{*''}(\zeta)}{F_p^{*'}(\zeta)}\right\}=pR\left\{\frac{\zeta^pF_1^{*''}(\zeta^p)}{F_1^{*'}(\zeta^p)}\right\}-(p-1)R\left\{\frac{\zeta^pF_1^{*'}(\zeta^p)}{F_1^*(\zeta^p)}\right\}+p. \tag{4}$$

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But from (2) it follows that

$$R\left\{\frac{\zeta^p F_1^{*''}(\zeta^p)}{F_1^{*'}(\zeta^p)}\right\} \leq \frac{2}{\rho^{2p}-1}, \tag{5}$$

and, since by virtue of the starlikeness of the complement of the image of the domain $\rho > 1$ we shall have $R\left\{\frac{\zeta F_1^{*'}(\zeta)}{F_1^*(\zeta)}\right\} > 0$ and the function $\Phi(z) = \frac{1}{z}\frac{F_1^{*'}(1/z)}{F_1^*(1/z)}$, $z = \frac{1}{\zeta}$, is regular in the disk $|z| < 1$, we may use the known estimate for functions regular in the disk with positive real part[4], from which we obtain $R\{\Phi(z)\} \geq \frac{1-|z|}{1+|z|} = \frac{\rho-1}{\rho+1}$, whence

$$\frac{\rho^p-1}{\rho^p+1} \leq R\left\{\frac{\zeta^p F_1^{*'}(\zeta^p)}{F_1^*(\zeta^p)}\right\}. \tag{6}$$

Moreover, for functions of Σ_p^* we have the estimate of I. E. Bazilevich[5]

$$\frac{(\rho^p-1)(\rho^p+1)^{2/p-1}}{\rho^2} \leq \left|F_p^{*'}(\zeta)\right|. \tag{7}$$

From (1), (4), (5), (6), and (7) we obtain (II).

The right-hand side of (II) is attained by the function

$$F_p^*(\zeta) = \frac{(\zeta^p+1)^{2/p}}{\zeta}$$

at the point $\zeta = \rho$.

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Note: Figure translations are in progress. See original paper for figures.

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