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Abstract

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MATHEMATICS

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TRANSFORMATION OF MONOTONE IRREDUCIBLE MAPPINGS INTO MONOTONE-OPEN ONES AND A MONOTONE-OPEN MAPPING OF A CUBE ONTO A CUBE OF HIGHER DIMENSION

(Presented by Academician P. S. Aleksandrov on 21 III 1957)

In the present paper we indicate a method by which, under certain conditions, one can obtain from a monotone irreducible mapping of a continuum X a monotone-open mapping of X. Since in many cases the problem of constructing a monotone irreducible mapping is simpler than the problem of constructing a monotone-open mapping, Theorems 1 and 2 give a method for constructing various examples of monotone-open mappings, in particular, monotone-open mappings that raise dimension. From Theorem 1 there follows a positive solution of P. S. Aleksandrov' s problem on the existence of an open mapping of a p-dimensional cube onto a q-dimensional one for $q > p \geq 3$. We note that a solution of this problem was announced by R. D. Anderson in 1953 (^{1}) and again in 1956 (^{2}), without indication of a method of construction, but the exposition of the proof has not yet appeared in print.

Here we shall describe briefly the idea of the proof of Theorems 1 and 2; a detailed exposition will be printed in the journal *Matematicheskii Sbornik.*

Recall that for an irreducible mapping f of a compactum X onto a compactum Y, in X there is everywhere dense the set E^{-1} of points of uniqueness, i.e. such points that $x = f^{-1}f(x)$; on E^{-1} the mapping f is a homeomorphism, and E^{-1}, as also $E = f(E^{-1})$, is a set of type G_δ.

Theorem 1. Let f be a monotone irreducible mapping of a continuum X onto a manifold (with boundary or without boundary) Y of dimension $n \geq 3$, and suppose that the intersection of every domain in Y with the set E of points of uniqueness of f is connected. Then, whatever the number $\varepsilon > 0$, there exists a continuous ε-shift Φ of the manifold Y onto itself such that the superposition $F = \Phi f$ is a monotone-open mapping of X onto Y.

Theorem 2. Let $f\colon X \to Y$ be a monotone irreducible mapping of a continuum X onto a locally connected continuum Y such that no connected open set (domain) U in Y is separated by a simple arc. If the intersection $E \cap U$ of any connected open set with the set of points of uniqueness is connected, then there exists a monotone mapping Φ of the continuum Y onto a continuum Z such that the superposition $F = \Phi f$ is a monotone-open mapping of the continuum X onto the continuum Z, and

$$\dim Z \geq \dim Y - 1.$$

We denote by $h(A, B)$ the deviation of the sets A and B, i.e. the upper bound of the distances from points of one of them to the other. By X_f we denote the continuous decomposition of the compactum X corresponding to the continuous mapping f.

The proof of Theorems 1 and 2 is based on the following two lemmas.

Lemma 1. Let f be a monotone irreducible mapping of a compactum X onto a locally connected compactum Y, no domain of which U is separated by a simple arc, and suppose that the intersection $E \cap U$ of the set of points of uniqueness—

is connected with any domain. Let

$$Y = \bigcup_{i=1}^{N} \Delta_i$$

be an ε-covering of multiplicity m by locally connected closures of domains, and suppose that $f^{-1}(\Delta_i)$ is contained in a neighborhood of diameter $< \varepsilon$ of an element of the decomposition $\xi_i \in X^f$. Then there exists a continuous decomposition Y^φ, having only a finite number of nondegenerate elements, each of which is the sum of not more than m simple arcs, contained in E and having one common end, and a covering

$$Y = \bigcup \delta_r$$

of multiplicity m, where δ_r contains only one point $f(\xi_i)$, $i = 1, \dots, N$, or a nondegenerate element $\eta_r \in Y^\varphi$; $f^{-1}(\delta_r)$ is a connected neighborhood of diameter $< \varepsilon$ for $\xi_i = \xi_i$ or for $\xi_r = f^{-1}(\eta_r)$; from $\delta_r \cap \delta_{r'} \neq \Lambda$ it follows that $h(\xi'_r, \xi'_{r'}) < \varepsilon$, and if

$$\bigcap_{k=1}^{p} \delta_{r_k} \neq \Lambda \quad \text{and} \quad \bigcup_{k=1}^{p} \delta_{r_k}$$

intersects Δ_{i_ν}, $\nu = 1, \dots, q$, then

$$\bigcap_{\nu=1}^{q} \Delta_{i_\nu} \neq \Lambda.$$

Lemma 2. Let f be a monotone irreducible mapping of a compactum X onto an n-dimensional locally connected compactum Y, no domain of which U is split by a simple arc, and let the intersection $U \cap E$ be connected. Let

$$Y = \bigcup_i \Delta_i$$

be a closed covering of multiplicity $m \leq n+1$, and suppose that $f^{-1}(\Delta_i)$ is a connected neighborhood of diameter $< \varepsilon$ for an element of the decomposition $\xi_i \in X^f$, and that from $\Delta_i \cap \Delta_j \neq \Lambda$ it follows that $h(\xi_i, \xi_j) < \varepsilon$. Then, whatever number $\varepsilon' > 0$ may be, there exists a continuous decomposition Y^ψ, having only a finite number of nondegenerate elements η_r, each of which is the sum of not more than m simple arcs having a common end y_r and contained in E except for the point y_r, and a covering

$$Y = \bigcup \delta_r$$

of multiplicity $\leq n+1$, where δ_r are locally connected closures of domains, each δ_r contains only one point $f(\xi_i)$ or an element η_r; $f^{-1}(\delta_r)$ is a neighborhood of diameter $< \varepsilon'$ for $\xi_r' = \xi_i$ or $\xi_r' = f^{-1}(\eta_r)$; from $\Delta_i \cap \delta_r \neq \Lambda$ it follows that $h(\xi_i, \xi_r') < 2\varepsilon$, and if

$$\bigcap_{k=1}^{p} \delta_{r_k} \neq \Lambda \quad \text{and} \quad \bigcup_{k=1}^{p} \delta_{r_k}$$

intersects Δ_{i_ν}, $\nu = 1, \dots, q$, then

$$\bigcap_{\nu=1}^{q} \Delta_{i_\nu} \neq \Lambda.$$

We indicate quite briefly the idea of the proof of Lemma 1. It is carried out by induction on the multiplicity m of the covering. If $m = 1$, then one may put $\delta_i = \Delta_i$, and all elements of the decomposition Y^φ are points. Let $m > 1$. We enclose each intersection $\Delta_{i_1} \cap \dots \cap \Delta_{i_m}$ in an open set whose closure $\pi_{i_1 \dots i_m}$ is locally connected and which is contained inside

$$\bigcup_{\nu=1}^{m} \Delta_{i_\nu},$$

and moreover so that distinct $\pi_{i_1 \dots i_m}$ do not intersect one another. Each $\pi_{i_1 \dots i_m}$ is the sum of a finite number of components π_μ. In each π_μ, for each $\pi_{i_1 \dots i_m}$, we choose an open set σ_μ so that

$$\Delta_{i_1} \cap \dots \cap \Delta_{i_m} \cap \pi_\mu \subset \sigma_\mu$$

and

$$Y' = Y \setminus \bigcup \sigma_\mu$$

is locally connected;

$$Y' = \bigcup_{i=1}^{N} (\Delta_i \cap Y')$$

has multiplicity $< m$, and for Y' there exists a covering

$$Y' = \bigcup \delta_r$$

satisfying the conditions of the lemma, and a mapping φ'. All nondegenerate elements of the decomposition $Y'^{\varphi'}$ will also be elements of the decomposition Y^{φ}.

The subsequent construction is made for each π_μ; therefore in what follows we omit the index μ. In each $\delta_r \cap \pi$ we choose a point of uniqueness y_r and draw in δ_r a tube τ_r, which is a neighborhood of diameter $< \varepsilon' < \varepsilon/2$ of some simple arc $l_r \subset E$, one end of which is y_r, while the other lies in δ_r, and such that, if η_r is an element of the decomposition $Y'^{\varphi'}$ whose neighborhood is δ_r, then $f^{-1}(\eta_r)$ is contained in the ε'-neighborhood of $f^{-1}(\tau_r)$ and $\tau_r \cap \eta_r = \Lambda$;

$$\tau_r = \bigcup_k u_{rk},$$

where u_{rk} is a neighborhood of diameter $< \varepsilon'$ of some point $y \in l_r$ and $\operatorname{diam} f^{-1}(u_{rk}) < \varepsilon'$, with $u_{rk} \cap u_{rk'} = \Lambda$ if $|k - k'| > 1$. The various tubes τ_r for all possible π_μ and δ_r may be chosen so that they do not intersect and do not split π_μ and δ_r. Then we put:

$$\theta = \pi \cup \bigcup_{\tau_r \cap \pi \neq \Lambda} \tau_r.$$

By construction,

$$\theta_\mu \cap \theta_{\mu'} = \Lambda.$$

From some point $y \in \pi \cap E$ we draw m simple arcs

$$l_\nu \subset (\pi \cup \tau_{i_\nu}) \cap E,$$

where $\tau_{i_\nu} \subset \delta_{i_\nu}$ and $\delta_{i_\nu} \supset f(\xi_{i_\nu})$. Let

$$L = \bigcup_{\nu=1}^{m} l_\nu;$$

this is an element of the decomposition Y^{φ}. To construct the remaining nondegenerate elements of Y^{φ} contained in θ, we choose in

$$\pi \cup \bigcup_{\nu=1}^{m} \tau_{i_\nu}$$

a tube

$$\tau = \bigcup_{k=1}^{k_0} u_k,$$

which satisfies the same conditions as the tubes τ_r; that is, a neighborhood of diameter $< \varepsilon'$ of a simple arc $l \subset E$, one end of which, $x \in u_1 \subset \pi$, and moreover

$$h(L, \tau) < \varepsilon', \qquad L \cap \tau = \Lambda.$$

Each nondegenerate element $\eta_\rho \in \theta^\varphi$, $\eta_\rho \neq L$, is a simple arc, and for some r and p

$$\eta_\rho \subset \tau_r \cup \tau \cup \bigcup_{k=1}^{p} u_k, \qquad p \leq k_0, \qquad \tau_r \cap \pi \neq \Lambda.$$

Here

$$\eta_\rho \subset E,$$

η_ρ meets all u_{rj}, one end of η_ρ lies in $\tau_r \setminus \pi$, and the other in u_p. The elements of the cover $\delta_\rho \subset \theta$ are constructed so that δ_ρ and $\delta_{\rho'}$ may intersect only if the ends of η_ρ and $\eta_{\rho'}$ lying in u_k and $u_{k'}$ are contained in summands u_k and $u_{k'}$ with $|k - k'| \leq 1$. For the construction, first the case is considered when the multiplicity of

$$\bigcup_r (\theta \cap \partial_r)$$

is equal to 1, and then the general case.

The construction and the proof are cumbersome and are carried out with the aid of additional inductive assumptions of a technical nature.

For the proof of Lemma 2 we choose the number ε' so that it is less than half the Lebesgue number for the given cover

$$Y = \bigcup \Delta_i,$$

and we choose for Y an ε'-cover

$$\bigcup \Delta'_r$$

of multiplicity $\leq n+1$ so that $f^{-1}(\Delta'_r)$ is a neighborhood of diameter $< \varepsilon'$ of some element $\xi_r \in X^f$. Each nondegenerate element $\eta_r \in Y^\psi$ is a sum of simple arcs l_{i_ν} having a common end

$$y_r = \bar{f}(\xi_r),$$

with

$$l_{i_\nu} \setminus y_r \subset E \cap (\Delta_{i_\nu} \cup \Delta'_r),$$

if

$$\Delta_{i_\nu} \cap \Delta'_r \neq \Lambda$$

and ξ_{i_ν} is contained in the ε'-neighborhood $f^{-1}(l_{i_\nu})$;

$$\delta_r \subset \Delta'_r \cup \bigcup_\nu \tau_{ri_\nu},$$

where the tubes τ_{ri_ν} are constructed in the same way as for Lemma 1. The construction is performed in each Δ_i first for Δ'_r intersecting $\Delta_i \supset f(\xi_i)$, and then, successively, for Δ'_r intersecting some Δ_ρ for which the construction has been performed.

For the proof of Theorem 1 it is easy to show that if Y is a manifold, then a simple arc lying in E can be constructed so that it has arbitrarily small neighborhoods homeomorphic to a ball. All tubes τ in Lemmas 1 and 2 are likewise homeomorphic to a ball. We show that in this case the mappings φ and ψ can be realized as ε-shifts of Y onto itself (for φ, for example, this is done by induction on m, on the basis of the fact that a small neighborhood θ can be mapped homeomorphically onto itself so that θ passes into π).

We choose a number ε and a convergent series

$$\varepsilon = \sum_{k=0}^{\infty} \varepsilon_k.$$

Having chosen an arbitrary ε_0-cover of Y and applying Lemmas 1 and 2 successively, we construct a sequence of continuous mappings:

$$f_{2k-1} = \varphi_k \psi_{k-1} \varphi_{k-1} \cdots \psi_1 \varphi_1 f;$$

$$f_{2k} = \psi_k \varphi_k \psi_{k-1} \cdots \psi_1 \varphi_1 f.$$

The mapping f_{2k+1} and the cover

$$Y = \bigcup_j \Delta_{2k+1,j}$$

are constructed by applying Lemma 1 to the mapping f_{2k} and the cover

$$Y = \bigcup_i \psi_k(\Delta_{2k,i}),$$

where

$\operatorname{diam} \Delta_{2k,i} < 2\varepsilon_{k-1}$, $\operatorname{diam} \psi_k(\Delta_{2k,i}) < \varepsilon_k$. The mapping $f_{2(k+1)}$ is constructed by applying Lemma 2 to the mapping f_{2k+1} and the covering $Y = \bigcup \varphi_{k+1}(\Delta_{2k+1,j})$, with $\operatorname{diam} \Delta_{2k+1,j} < 2\varepsilon_k$ and $\operatorname{diam} \varphi_{k+1}(\Delta_{2k+1,j}) < \varepsilon_k$. The mappings φ_k and ψ_k are ε_k-shifts of Y onto itself. Therefore the sequence of mappings f_m converges uniformly to a continuous mapping F of the continuum X onto the manifold Y. Putting

$$\Phi = \lim_{k\to\infty} \varphi_k \psi_k \cdots \varphi_1 \psi_1,$$

we obtain

$$F = \Phi f.$$

We show that the mapping F is monotone-open.

Corollary 1. *There exists a monotone-open mapping of the p-dimensional cube C_p onto the q-dimensional cube C_q, $q > p \geqslant 3$.*

It suffices to consider the case $p = 3$, $q = 4$, since the general case is reduced to it. In (^{3},^{4}) we constructed a monotone irreducible mapping f of the three-dimensional cube C_3 onto the four-dimensional cube C_4. It is easy to show that, for the mapping f, the set of multiple points $C_4 \setminus E$ (of type F_σ) has dimension 2; consequently, E is connected in every domain, and the conditions of Theorem 1 are satisfied; hence there exists an ε-shift Φ of the cube C_4 onto itself such that Φf is a monotone-open mapping of the cube C_3 onto the cube C_4.

For the proof of Theorem 2 we construct a continuous decomposition Y^Φ, again successively applying Lemmas 1 and 2. It is easy to show that all elements of the continuous decomposition Y^Φ are one-dimensional continua. Consequently,

$$\dim \Phi(Y) \geqslant \dim Y - 1.$$

Theorem 2 is also true in the case when $n = \infty$; in this case the multiplicities of the coverings increase to infinity.

Applying Theorem 2 to the example constructed by us of a monotone irreducible mapping of a cube onto a continuum, in a neighborhood of every point of which there is contained a topological Hilbert parallelepiped (5), we find:

Corollary 2. *There exists a monotone-open mapping of the three-dimensional cube onto a continuum, in every neighborhood of each point of which there is contained a topological Hilbert parallelepiped.*

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