G. A. ZAITSEV

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Abstract

Full Text

G. A. ZAITSEV

ON THE QUESTION OF THE BASIC RELATIVISTICALLY INVARIANT EQUATION FOR A PARTICLE WITH SPIN 1/2

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Since the magnetic moment of a particle, as a rule, differs significantly from the corresponding magneton, and only for the electron and positron does the difference between them turn out to be small, the Dirac equation is not applicable to arbitrary particles with spin 1/2. Therefore we shall reconsider the problem of finding a relativistically invariant equation for a particle with spin 1/2, charge e, mass m_0, and magnetic moment

$$\mu_0 = \frac{e\hbar}{2m_0c}(1+\delta)$$

(where $\delta \neq 0$), situated in an external electromagnetic field.

In the nonrelativistic approximation the equation for a particle with spin 1/2 may be written in the form

$$\left[-i\hbar\frac{\partial}{\partial t} + \frac{1}{2m_0}\sum_{k=1}^{3}\left(-\hbar i\frac{\partial}{\partial x^k} - \frac{e}{c}A_k\right)^2 + e\varphi\right]\xi_0 - \frac{e\hbar}{2m_0c}(1+\delta)\sum_{k=1}^{3}H_k\sigma_k\xi_0 = 0 \qquad (1)$$

(here ξ_0 is a spinor of rank 1; σ_k are the Pauli matrices; A_k are the components of the vector potential).

As is known, in studying the behavior of a particle in a magnetic field, equation (1) usually serves to determine the magnetic moment. For the electron (if the quantity δ is neglected) equation (1) is the usual Pauli equation.

We shall proceed from the fact that the state of a particle with spin 1/2 can be characterized by specifying one real spinor ψ, whose transformation law is known for the full Lorentz group ([1,2]). Both sides of the relations

$$-\hbar i\frac{\partial\xi}{\partial x^\alpha} = p_\alpha\xi$$

or, equivalently,

$$-\hbar I\frac{\partial\psi}{\partial x^\alpha} = p_\alpha\psi$$

(where, according to ([1,2]), we put $\xi \leftrightarrow \psi,\ i \leftrightarrow I,\ x^4 = ct,\ \alpha = 1,2,3,4$) must transform in the same way, and the p_α are the components of a four-dimensional vector. Therefore $\hbar$ must here be regarded not as a scalar (as is usually assumed), but as a pseudoscalar, changing sign under space-time reflections and remaining unchanged under four-dimensional rotations (so that the numerical value of Planck' s constant is equal to the absolute value of the pseudoscalar), while the matrix I must be equal to $\pm J$. Taking, for definiteness, the plus sign, we find that the expressions for the matrices R^α are given by the first row of Table 1 of ([2]). Using this and putting $\xi_0 \leftrightarrow \psi$, we may rewrite equation (1) in the form

$$\left[-\hbar J\frac{\partial}{\partial t}+\frac{1}{2m_0}\sum_{k=1}^{3}\left(-\hbar J\frac{\partial}{\partial x^k}-\frac{e}{c}A_k\right)^2+e\varphi\right]\psi_0-\frac{e\hbar}{2m_0c}(1+\delta)R_4\sum_k R_k^{kH}\psi_0=0. \tag{2}$$

In order to make equation (2) relativistically invariant, it is necessary to replace the operator

$$-\hbar J\frac{\partial}{\partial t}-eA_4+\frac{1}{2m_0}\sum_k\left(\hbar J\frac{\partial}{\partial x^k}+\frac{e}{c}A_k\right)^2 \qquad (A_4=-\varphi)$$

by

$$\frac{1}{2m_0}\left[-\left(\hbar J\frac{\partial}{\partial x^4}+\frac{e}{c}A_4\right)^2+\sum_{k=1}^{3}\left(\hbar J\frac{\partial}{\partial x^k}+\frac{e}{c}A_k\right)^2+m_0^2c^2\right],$$

and $R_4\sum_k R_k^{kH}$ by

$$R_4\sum_k R_k^{kH}-R\sum_k R_k^{kE}=-J\frac{1}{2}F_{\alpha\beta}R^\alpha R^\beta$$

($F_{\alpha\beta}$ are the components of the electromagnetic-field tensor).

Then (2) becomes

$$\left[-\left(\hbar J\frac{\partial}{\partial x^4}+\frac{e}{c}A_4\right)^2+\sum_k\left(\hbar J\frac{\partial}{\partial x^k}+\frac{e}{c}A_k\right)^2+m_0^2c^2+\frac{e\hbar}{c}(1+\delta)J\frac{1}{2}F_{\alpha\beta}R^\alpha R^\beta\right]\psi=0 \tag{3}$$

(where ψ is a real spinor), and the invariance of this equation with respect to the full Lorentz group follows directly from the properties of real spinors ([2]). In order to obtain equation (2) from (3), one must put

$$\psi=\left[\exp\left(-J\frac{m_0c^2t}{\hbar}\right)\right]\psi_0$$

and neglect the quantities

$$\frac{m_0\hbar^2}{2c^2}\frac{\partial^2\psi_0}{\partial t^2}, \qquad \frac{e^2\varphi^2 m_0}{2c^2}\psi_0, \qquad \frac{m_0 e\hbar}{2c^2}\left(J\frac{\partial}{\partial t}\varphi + J\varphi\frac{\partial}{\partial t}\right)\psi_0,$$

as well as discard the term

$$\frac{e\hbar}{c}(1+\delta)JR_4\sum R_k^{kE}.$$

Let us consider the special case of equation (3) when $\mu_0 = e\hbar/2m_0c$. Denoting ψ in this case by $\psi_{(1)}$ and introducing a second real spinor $\psi_{(2)}$, we can, for $\delta = 0$, obtain (3) from the system of relativistically invariant first-order differential equations:

$$R^\alpha\frac{\partial}{\partial x^\alpha}\psi_{(1)} + \frac{e}{\hbar c}JR^\alpha A_\alpha\psi_{(1)} = \frac{m_0c}{\hbar}\psi_{(2)}, \tag{4}$$

$$R^\alpha\frac{\partial}{\partial x^\alpha}\psi_{(2)} - \frac{e}{\hbar c}JR^\alpha A_\alpha\psi_{(2)} = \frac{m_0c}{\hbar}\psi_{(1)}. \tag{5}$$

For this it is sufficient to multiply (4) on the left by the operator

$$R^\alpha\frac{\partial}{\partial x^\alpha} - \frac{e}{\hbar c}JR^\alpha A_\alpha.$$

Introducing, instead of $\psi_{(1)}$ and $\psi_{(2)}$, a column with complex elements

$$\tilde{\psi} = \frac{1}{\sqrt{2}}\left[(\psi_{(2)} + \psi_{(1)}) + iJ(\psi_{(2)} - \psi_{(1)})\right],$$

we can rewrite (4) and (5) in the form of a single equation

$$R^\alpha\left(\frac{\partial}{\partial x^\alpha} - i\frac{e}{\hbar c}A_\alpha\right)\tilde{\psi} = \frac{m_0c}{\hbar}\tilde{\psi}. \tag{6}$$

But this is the Dirac equation with a special choice of matrices:

$$\gamma_k = R^k, \qquad \gamma_4 = iR^4$$

(where, however, since $\hbar$ is a pseudoscalar, under reflection $\tilde{\psi}$ is transformed not quite in the usual way). Therefore all solutions of the Dirac equation with the corresponding choice of matrices will at the same time be solutions of (3) for $\delta = 0$.

From the point of view considered by us, the original fundamental equation is (3), and not the Dirac equation (6). Differential

equations of the first order are introduced only for the convenience of solving the fundamental equation (analogously to the method of factorization of (3)); they have no independent significance. In the general case, for $\delta \neq 0$, (3) can no longer be obtained from a system of first-order equations.

In conclusion, let us write the fundamental equation (3) with the aid of complex quantities (putting $\xi = \binom{\psi_1 + i\psi_3}{\psi_2 + i\psi_4}$, $i \leftrightarrow J$, etc.):

$$\left(\hbar i \frac{\partial}{\partial t} - e\varphi\right)^2 - c^2 \sum_k \left(\hbar i \frac{\partial}{\partial x^k} + \frac{e}{c} A_k\right)^2 - m_0^2 c^4 -$$

$$-e\hbar c(1+\delta)\left(-\sum_k H_k \sigma_k + i \sum_k E_k \sigma_k\right) \xi = 0. \qquad (7)$$

In this form it has an especially simple appearance and is convenient to use in solving concrete physical problems, whereas in general investigations it is more expedient to use the notation (3). In particular, for $\delta = 0$, (7) looks considerably simpler than the second-order equation obtained from the Dirac equation in the usual way. A characteristic feature of (7) is also that, on the left, an operator acts on ξ which is not Hermitian-conjugate (for $E_k \neq 0$), so that the postulate of quantum mechanics concerning the Hermiticity of operators should be regarded only as an approximation. Probably it is precisely because of this last feature that, in attempts to generalize the Pauli equation, equations of the type (7) were not considered.

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Note: Figure translations are in progress. See original paper for figures.