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Let us take the series

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Abstract

Full Text

MATHEMATICS

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ON LIMIT FUNCTIONS OF A TRIGONOMETRIC SERIES

Let us take the series

$$\sum_{n=0}^{\infty} u_n(x), \tag{1}$$

whose terms are measurable functions, finite almost everywhere on some segment $[a, b]$. Put

$$Q_n(x) = \sum_{\nu=0}^{n} u_\nu(x) \tag{2}$$

and introduce the following definition.

Definition 1. A function $\varphi(x) \equiv \varphi(x, E)$, defined almost everywhere on some set $E \subset [a, b]^*$ of positive measure, will be called a **limit function** of the series (1) if there exists an increasing sequence of natural numbers ρ_k, $k = 1, 2, \dots$, such that

$$\lim_{k\to\infty} Q_{\rho_k}(x) = \varphi(x) \tag{3}$$

almost everywhere on E^{**}.

The aim of the present note is to study the set of all limit functions of an arbitrary trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{4}$$

Let us take a set $M = \{\varphi(x, E)\}$ of measurable functions $\varphi(x, E)$, each of which is defined on some set $E \subset [-\pi, \pi]$ of positive measure. (The sets E may be different for different functions $\varphi(x, E) \in M$.) One can obtain a necessary and sufficient condition for the set M to be the set of all limit functions of some trigonometric series. To formulate this condition, we introduce some definitions.

Definition 2. Let us take a sequence of functions

$$\varphi_n(x, E_n) \qquad (n = 1, 2, ...), \tag{5}$$

* When we say that a measurable function is defined almost everywhere on some measurable set, we do not exclude the possibility that this function is equal to $+\infty$ or $-\infty$ on sets of positive measure.

** From this definition follows the assertion:

If $\varphi(x, E)$ is a limit function of the series (1) and if $E' \subset E$, $\operatorname{mes} E' > 0$, and $\varphi(x, E') = \varphi(x, E)$ almost everywhere on E', then $\varphi(x, E')$ is also a limit function of the series (1).

In this case, if E' does not coincide with E, then we shall regard $\varphi(x, E)$ and $\varphi(x, E')$ as different functions.

each of which is defined almost everywhere on the corresponding set E_n. Put

$$E = \lim_{n\to\infty} E_n \tag{6}$$

and suppose that the set E and some other set E' satisfy the conditions

$$\operatorname{mes} E > 0, \qquad \operatorname{mes} E' > 0, \qquad \operatorname{mes}(E' - E) = 0. \tag{7}$$

We shall say that a function $\varphi(x, E')$, defined almost everywhere on the set E', is a **limit element in the broad sense** of the sequence (5), if

$$\lim_{n\to\infty} \varphi_n(x, E_n) = \varphi(x, E') \tag{8}$$

almost everywhere on E'.*

It is easy to see that a limit element in the broad sense of the sequence (5) is not uniquely determined. Indeed, if $\varphi(x, E')$ is a limit element in the broad sense of the sequence (5) and $\operatorname{mes} E'' > 0$, $\operatorname{mes}(E'' - E') = 0$, then the function $\varphi(x, E'')$, equal to $\varphi(x, E')$ almost everywhere on E'', will also be a limit element in the broad sense of the sequence (5).

Definition 3. Take some set $M = \{\varphi(x, E)\}$ of functions $\varphi(x, E)$, each of which is defined on the corresponding set E, $\operatorname{mes} E > 0$. We shall call a function $f(x, E')$, defined almost everywhere on a set E' of positive measure, a **limit**

element in the broad sense of the set M, if there exists a sequence of functions $\varphi_n(x, E_n)$, belonging to the set M, for which $f(x, E')$ is a limit element in the broad sense.

In this definition the functions $\varphi_n(x, E_n)$ need not be distinct for different n. In such a case, as is easy to show, any function $\varphi(x, E)$ of the set $M = \{\varphi(x, E)\}$ is a limit element in the broad sense of this set.

Definition 4. Let the set $M = \{\varphi(x, E)\}$ satisfy the same conditions as in Definition 3. We shall call this set **closed in the narrow sense** if it contains all its limit elements in the broad sense.

The following theorem holds:

Theorem 1. *Let $M = \{\varphi(x, E)\}$ be a set of measurable functions, each of which is defined almost everywhere on some set E of positive measure, lying on $[-\pi, \pi]$.*

In order that M be the set of all limit functions of the trigonometric series (4), it is necessary and sufficient that this set be closed in the narrow sense.

The first part of Theorem 1, i.e. the necessity of the condition appearing in its formulation, may be obtained from the following theorem:

Theorem 2. *Let the terms of the series (1) be measurable and finite almost everywhere on some segment $[a, b]$, and let $F(x)$ and $G(x)$ be, respectively, the upper and lower limits in measure on $[a, b]$ of this series**.*

* It makes sense to speak of the limit of the function $\varphi_n(x, E_n)$ as $n \to \infty$ almost everywhere on E', since on the basis of (6) and (7), at every point $x \in E'$, except possibly for a set of measure zero, the function $\varphi_n(x, E_n)$ is defined for all sufficiently large values of n. It is clear in what sense the equality (8) should be understood if $\varphi_n(x, E_n)$ takes the values $+\infty$ or $-\infty$ ($\varphi(x, E')$ may also take the values $+\infty$ or $-\infty$).

** The definition of the upper and lower limits in measure on $[a, b]$ of the series (1) is given in (1) (Definition 5, p. 779). See also (2), § 1, p. 4.

Then the set $M = \{\varphi(x, E)\}$ of all limit functions of the series (1) satisfies the following conditions:

α) M is closed in the narrow sense;

β) if $\varphi(x, E) \in M$, then $\varphi(x, E)$ is measurable on E and

$$G(x) \leqslant \varphi(x, E) \leqslant F(x)$$

almost everywhere on this set;

γ) if $\varphi(x, E) \in M$ and if

$$E_0 = E + E[F(x) = G(x)]^*,$$

then the function $\varphi_0(x, E_0)$, defined by the equality

$$\varphi_0(x, E_0) = \begin{cases} \varphi(x, E) & (x \in E), \\ F(x) & (x \in E_0 - E), \end{cases} \tag{9}$$

also belongs to the set M**.

The second part of Theorem 1, i.e. the sufficiency of the condition contained in it, follows from the following theorem:

Theorem 3. Let $M = \{\varphi(x, E)\}$ be a nonempty set of functions $\varphi(x, E)$, each of which is defined on the corresponding set E, where

$$\operatorname{mes} E > 0, \qquad E \subset [-\pi, \pi].$$

Suppose that the set M satisfies conditions α, β, and γ of Theorem 2, where $F(x)$ and $G(x)$ are measurable functions, defined almost everywhere on $[-\pi, \pi]$ and satisfying the inequality

$$G(x) \leqslant F(x)$$

almost everywhere on this segment***.

In that case there exists a trigonometric series (4) possessing the following properties:

1) M is the set of all limit functions of the series (4);
2) $F(x)$ and $G(x)$ are, respectively, the upper and lower limits in measure on $[-\pi, \pi]$ of the series (4);
3) $\lim\limits_{n\to\infty} a_n = 0, \quad \lim\limits_{n\to\infty} b_n = 0.$

Theorems 2, 3, 4, and 5 in ([1]) are obtained as corollaries of the theorems formulated in the present note.

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REFERENCES

[1] D. Menshov, DAN, **106**, No. 5, 777 (1956). [2] D. Menshov, Tr. Mat. Inst. im. V. A. Steklova AN SSSR, **32**, 3 (1950).

* By $E[F(x) = G(x)]$ we denote the set of all points x in $[a, b]$ at which $F(x) = G(x)$.

** In equality (9) we define the functions $\varphi(x, E)$ and $F(x)$ arbitrarily on a set of measure zero where they were not originally defined.

*** We suppose that conditions α, β, and γ are formulated for the segment $[a, b] = [-\pi, \pi]$.

Note: Figure translations are in progress. See original paper for figures.