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Abstract

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EMBEDDING THEOREMS FOR A SPACE WITH A METRIC DEGENERATING ON A RECTILINEAR PART OF THE BOUNDARY OF THE DOMAIN

(Presented by Academician S. L. Sobolev on 16 XI 1956)

Let D be a finite domain located in the upper half-plane and having a part of the boundary Γ_0 on the Ox axis. We shall denote the remaining part of the boundary by Γ_1. In this case the complete boundary is $\Gamma = \Gamma_0 \cup \Gamma_1$. We assume that Γ_1 is such that the embedding theorems of S. L. Sobolev ([1]) hold for it.

Let Ω^0 be the manifold of all functions continuous in the domain D, having bounded piecewise-continuous second derivatives and vanishing in some boundary strip of the domain D (their own for each function). On the set of functions $u^0 \in \Omega^0$ we define a gradient-type operator:

$$Gu^0 = \left(\frac{\partial^2 u^0}{\partial x^2}, \frac{\partial^2 u^0}{\partial x\, \partial y}, \frac{\partial^2 u^0}{\partial y^2} \right).$$

Thus, the operator G assigns to the function $u^0 \in \Omega^0$ the system of all its partial derivatives of second order. The manifold composed of the elements Gu^0, where $u^0 \in \Omega^0$, will be denoted by R^0.

Introduce in R^0 the scalar product by the formula:

$$\{Gu^0, Gv^0\} = \iint_D \left[a_{1111} \frac{\partial^2 u^0}{\partial x^2} \frac{\partial^2 v^0}{\partial y^2} + a_{1212} \frac{\partial^2 u^0}{\partial x\, \partial y} \frac{\partial^2 v^0}{\partial x\, \partial y} + a_{2222} \frac{\partial^2 u^0}{\partial y^2} \frac{\partial^2 v^0}{\partial y^2} + \right.$$
$$+ \frac{1}{2} a_{1112} \frac{\partial^2 u^0}{\partial x^2} \frac{\partial^2 v^0}{\partial x\, \partial y} + \frac{1}{2} a_{1112} \frac{\partial^2 u^0}{\partial x\, \partial y} \frac{\partial^2 v^0}{\partial x^2} + \frac{1}{2} a_{1222} \frac{\partial^2 u^0}{\partial y^2} \frac{\partial^2 v^0}{\partial x\, \partial y} +$$
$$\left. + \frac{1}{2} a_{1222} \frac{\partial^2 u^0}{\partial x\, \partial y} \frac{\partial^2 v^0}{\partial y^2} + \frac{1}{2} a_{1122} \frac{\partial^2 u^0}{\partial x^2} \frac{\partial^2 v^0}{\partial y^2} + \frac{1}{2} a_{1122} \frac{\partial^2 u^0}{\partial y^2} \frac{\partial^2 v^0}{\partial x^2} \right] dx\, dy, \quad (1)$$

where the following restrictions are imposed on the coefficients $a_{1111}, a_{1112}, a_{1122}, a_{1212}, a_{1222}, a_{2222}$:

1) all of them are continuous in the closed domain $\overline{D} = D \cup \Gamma$;

2) either $a_{1111} \to 0$ as $y \to 0$, or $a_{2222} \to 0$ as $y \to 0$;

3) for any real numbers $\xi_{11}, \xi_{12}, \xi_{22}$ such that $\xi_{11}^2+\xi_{12}^2+\xi_{22}^2>0$, the quadratic form

$$B(\xi_{11}, \xi_{12}, \xi_{22}; x, y) \equiv a_{1111}\xi_{11}^2+a_{1212}\xi_{12}^2+a_{2222}\xi_{22}^2+a_{1112}\xi_{11}\xi_{12}+a_{1222}\xi_{12}\xi_{22}+a_{1122}\xi_{11}\xi_{12}>0 \tag{2}$$

everywhere in the domain $\overline{D}$, with the equality sign attained only at points $(x, y) \in \Gamma_0$. We introduce the metric in R^0 in the usual way, through the scalar product, and shall call it "metric (1)."

Denote by $\dot{R}$ the closure of the space R^0 in metric (1). By virtue of the equivalence of metric (1) and the metric $W_2^{(2)}$ in $D^\delta = D \cap (y \geq \delta)$, $\delta > 0$,

and also on the basis of the inequality

$$\iint_{D^\delta} (u^0)^2\,dx\,dy \leq C^2 \iint_{D^\delta} \left[\left(\frac{\partial^2 u^0}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 u^0}{\partial x\,\partial y}\right)^2 + \left(\frac{\partial^2 u^0}{\partial y^2}\right)^2\right] dx\,dy, \tag{3}$$

which follows from the embedding theorems of S. L. Sobolev ([1]), it follows that in every domain D^δ the functions $u_n^0 \in \Omega^0$ converge in the mean to a certain function u, which has generalized second derivatives in this domain. Hence it follows that the element $g \in \dot{R}$ is equal to the system of generalized derivatives of second order (in the sense of S. L. Sobolev) of the function u in the domain D^δ.

Since the conclusion is valid for any domain D^δ, where $\delta > 0$ is arbitrary, it follows from this that the element $g \in \dot{R}$ is equal to the system of generalized partial derivatives of second order of the function u in the domain D, i.e.,

$$g = Gu = \left(\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x\,\partial y}, \frac{\partial^2 u}{\partial y^2}\right).$$

For $Gu,\ Gv \in \dot{R}$, the scalar product $\{Gu, Gv\}$ can also be computed by formula (1), where the integral must be understood in the sense of Lebesgue. Denote by $\dot{\Omega}$ the manifold of all functions u obtained as a result of the above-described completion process. Obviously, $G\dot{\Omega} = \dot{R}$, and consequently $\dot{\Omega}$ is the domain of definition of the gradient-type operator G.

Sometimes we shall assume that

$$c^2 y^\alpha \leq a_{2222} \leq C^2 y^\alpha; \tag{4}$$

$$\bar{c}^2 y^\beta \leq a_{1212} \leq \bar{C}^2 y^\beta; \tag{5}$$

and also that, for any real $\xi_{11}, \xi_{12}, \xi_{22}$,

$$0 \leq y^{\alpha}\xi_{22}^2 \leq C_1^2 B(\xi_{11}, \xi_{12}, \xi_{22};\, x, y); \tag{6}$$

$$0 \leq y^{\beta}\xi_{12}^2 \leq C_2^2 B(\xi_{11}, \xi_{12}, \xi_{22};\, x, y); \tag{7}$$

$$0 \leq a_{1111}\xi_{11}^2 \leq C_3^2 B(\xi_{11}, \xi_{12}, \xi_{22};\, x, y). \tag{8}$$

Theorem 1. 1) On the part of the boundary $\Gamma_1^\delta = \Gamma_1 \cap (y \geq \delta)$, where $\delta > 0$ is arbitrary, every function from $\dot\Omega$ takes, in the mean, the value zero together with its first derivatives.

2) If condition (6) is satisfied for $0 \leq \alpha < 1$ and (7) for $0 \leq \beta < 1$, then on Γ_0 every function from $\dot\Omega$ takes, in the mean, the value zero together with its first derivatives.

3) If condition (7) is satisfied for $0 \leq \beta < 1$, then on Γ_0 every function from $\dot\Omega$ takes, in the mean, the value zero together with the first derivative with respect to x.

4) If condition (6) is satisfied for $0 \leq \alpha < 1$, then on Γ_0 every function from $\dot\Omega$ takes, in the mean, the value zero together with the first derivative with respect to y.

5) If condition (6) is satisfied for $1 \leq \alpha < 3$, then every function $u \in \dot\Omega$ takes, in the mean, the value zero on Γ_0.

6) If for $\alpha \geq 1$, β arbitrary, or for $\beta \geq 1$, α arbitrary, conditions (4), (5), (6), (7), and (8) are satisfied, then any function $\varphi(x, y)$ that has bounded piecewise-continuous second derivatives in D^δ, where $\delta > 0$ is arbitrary, and vanishes together with its first derivatives on Γ_1, moreover:

a) $\{G\varphi, G\varphi\} < +\infty$;

b) $|\varphi| \leq C^2 y^{\frac{3-\alpha}{2}}$ for $\alpha \neq 3$; $|\varphi| \leq c^2 |\ln y|^{1/2}$ for $\alpha = 3$;

c) $\left|\dfrac{\partial\varphi}{\partial x}\right| \leq C_1^2 y^{\frac{1-\beta}{2}}$ for $\beta \neq 1$; $\left|\dfrac{\partial\varphi}{\partial x}\right| \leq C_1^2 |\ln y|^{1/2}$ for $\beta = 1$;

c) $\left|\dfrac{\partial\varphi}{\partial y}\right| \leq C_2^2 y^{\frac{1-\alpha}{2}}$ for $\alpha \neq 1$; $\left|\dfrac{\partial\varphi}{\partial y}\right| \leq C_2^2 |\ln y|^{1/2}$ for $\alpha = 1$,

belongs to $\dot\Omega$.

The proof of item 1 follows from the embedding theorems of S. L. Sobolev ([1]) and the equivalence of the metrics (1) and $W_2^{(2)}$ in D^δ.

The proofs of items 2, 3, 4, and 5 do not essentially differ from one another. We shall therefore restrict ourselves to proving item 5 for $1 < \alpha < 3$. Extend the function $u \in \dot{\Omega}$ by zero into the region $(y > 0) \setminus D$, and let the number A be so large that the point (x, A) lies outside the domain D.

For functions $u \in \Omega^0$ the following estimate holds:

$$[u(x,y)]^2 \leq C^2 y^{3-\alpha} \int_0^A y^\alpha \left(\frac{\partial^2 u}{\partial y^2}\right)^2 dy. \tag{9}$$

Integrating both sides of inequality (9) over $\Gamma_h = D \cap (y = h)$ and using (6), we shall have:

$$\int_{\Gamma_h} u^2 \, d\Gamma_h \leq C_1^2 h^{3-\alpha} \{Gu, Gu\}. \tag{10}$$

By means of passage to the limit we verify the validity of estimate (10) for almost all h for any function $u \in \dot{\Omega}$. This implies item 5 of the theorem for $1 < \alpha < 3$.

For the proof of item 6 of the theorem, introduce the function

$$\chi_\delta = \chi_\delta(y) = \begin{cases} 0, & 0 \leq y < \delta, \\ \{1 - [(\ln|\ln y|)^\varepsilon - (\ln|\ln \delta_1|)^\varepsilon]^2\}^2, & \delta \leq y \leq \delta_1, \\ 1, & y > \delta_1, \end{cases}$$

where

$$(\ln|\ln \delta|)^\varepsilon - (\ln|\ln \delta_1|)^\varepsilon = 1, \qquad 0 < \varepsilon < 1/2.$$

It is obvious that the function $\varphi_\delta = \varphi\chi_\delta \in \dot{\Omega}$. It is shown that $G\varphi_\delta$ converges in the metric (1) to $G\varphi$. Hence it will follow that $G\varphi \in \dot{R}$, $\varphi \in \dot{\Omega}$.

It follows from the theorem that, when conditions (6) and (7) are satisfied, for $0 \leq \alpha < 1,\ 0 \leq \beta < 1$, $\dot{\Omega}$ contains functions that vanish together with their first derivatives on the entire boundary $\Gamma = \Gamma_1 \cup \Gamma_0$. When conditions (4), (5), (6), (7), and (8) are satisfied, $\dot{\Omega}$ contains functions: in the case $\alpha \geq 1,\ 0 \leq \beta < 1$, equal to zero on Γ together with the first derivative with respect to x; the derivatives with respect to y of these functions vanish only on Γ_1, while on Γ_0 they may tend to infinity; in the case $0 \leq \alpha < 1,\ \beta \geq 1$, equal to zero on Γ together with the derivative with respect to y; the derivative with respect to x of these functions vanishes on Γ_1, while on Γ_0 it may or may not vanish; in the case $1 \leq \alpha < 3,\ \beta \geq 1$, equal to zero on Γ, whose first derivatives with respect to x and y vanish on Γ_1; the first derivatives on Γ_0 may tend to infinity; in the case $\alpha \geq 3,\ \beta \geq 1$, equal to zero on Γ_1 together with the first derivatives and tending to infinity together with the first derivatives as $y \to 0$.

Vanishing is understood in the mean, in the sense of S. L. Sobolev.

Theorem 2. If inequalities (6) and (7) are satisfied, then for functions $u \in \dot{\Omega}$ the following estimates hold:

$$\iint_D \sigma_0(x,y) u^2(x,y) \, dx \, dy \leq C^2 \{Gu, Gu\};$$

$$\iint_D \sigma_1(x,y)\left(\frac{\partial u}{\partial y}\right)^2 dx\,dy \le C_1^2\{Gu,Gu\};$$

$$\iint_D \sigma_2(x,y)\left(\frac{\partial u}{\partial x}\right)^2 dx\,dy \le C_2^2\{Gu,Gu\},$$

where C^2, C_1^2, C_2^2 do not depend on the function u; $\sigma_i(x,y)$, $i = 0,1,2$, are sufficiently smooth functions; $\sigma_i(x,y) > 0$ for $y > 0$, and

$$\sigma_0(x,y) = \begin{cases} O\left(y^{\alpha-4}|\ln y|^{-1-\varepsilon_0}\right), & \text{for } \alpha \neq 1,\ \alpha < 3,\ \beta \text{ arbitrary;} \\ O\left(y^{-3}|\ln y|^{-2-\varepsilon_0}\right), & \text{for } \alpha = 1,\ \beta \text{ arbitrary;} \\ O\left(y^{\beta-2}|\ln y|^{-1-\varepsilon_0}\right), & \text{for } \alpha \ge 3,\ \beta < 1; \\ O\left(y^{-1}|\ln y|^{-2-\varepsilon_0}\right), & \text{for } \alpha \ge 3;\ \beta = 1;\ \alpha = 3,\ \beta \ge 1; \\ O\left(y^{\alpha-4}|\ln y|^{-1-\varepsilon_0}\right) \text{ or } O\left(y^{\beta-2}|\ln y|^{-1-\varepsilon_0}\right), & \text{for } \alpha > 3,\ \beta > 1; \end{cases}$$

$$\sigma_1(x,y) = \begin{cases} O\left(y^{\alpha-2}|\ln y|^{-1-\varepsilon_0}\right), & \text{for } \alpha \neq 1; \\ O\left(y^{-1}|\ln y|^{-2-\varepsilon_0}\right), & \text{for } \alpha = 1; \end{cases}$$

$$\sigma_2(x,y) = \begin{cases} O\left(y^{\beta-2}|\ln y|^{-1-\varepsilon_0}\right), & \text{for } \beta \neq 1; \\ O\left(y^{-1}|\ln y|^{-2-\varepsilon_0}\right), & \text{for } \beta = 1, \end{cases} \qquad \varepsilon_0 > 0 \text{ and arbitrary.}$$

It follows from Theorem 2 that, in the case $\alpha < 4$, for any $\beta \ge 0$, all functions $u \in \dot{\Omega}$, under condition (6), are square-summable over the domain D; if $\alpha < 2$, then, under condition (6), the first derivatives with respect to y are also square-summable over the domain D; if $\beta < 2$, for any $\alpha \ge 0$, then all functions $u \in \dot{\Omega}$, under condition (7), are square-summable over the domain D together with the first derivatives with respect to x.

We note that there are examples showing that the order of the weights $\sigma_i(x,y)$ is exact with the degree of accuracy indicated in [2].

All the results of the present note are easily generalized to the case when derivatives of order m participate in the scalar product, and the coefficients and derivatives depend on n variables.

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References

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2. M. I. Vishik, *Matem. sborn.*, **35** (77), 513 (1954).

Note: Figure translations are in progress. See original paper for figures.