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Abstract

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MATHEMATICS

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ON THE SPECTRAL FUNCTIONS OF A DIFFERENTIAL OPERATOR OF EVEN ORDER

(Presented by Academician A. N. Kolmogorov on 23 I 1957)

1. Let $l[y]$ be a self-adjoint ordinary differential expression of even order with real coefficients, given on the interval (a, b), whose endpoints are regular or singular—indifferently. As is known, the expression $l[y]$ generates in the Hilbert space $L^2(a, b)$ a symmetric differential operator L with minimal domain of definition*. On the basis of the formula, established in the author’ s note (^{4}), for the generalized resolvents of the operator L, the present paper gives an effective construction of all spectral functions of this operator. Under the assumption that both endpoints of the interval (a, b) are regular, a class of boundary-value problems with boundary conditions depending on a parameter is investigated, and an expansion formula in eigenfunctions of such problems is established.

2. In (^{4}) it is shown that every generalized resolvent R_λ ($\operatorname{Im}\lambda \neq 0$) of the operator L is an integral operator, and the corresponding formula for its kernel $K(x, s; \lambda)$ is obtained. Denote by $K^{[j,k]}(x, s; \lambda)$ the so-called quasi-derivatives of this kernel of order j with respect to x and of order k with respect to s $(j, k = 0, 1, \dots, 2n - 1)$.

Put

$$m_{jk}(\lambda) = \frac{1}{2}\left[K^{[j-1,k-1]}(x_0, x_0 - 0; \lambda) + K^{[j-1,k-1]}(x_0, x_0 + 0; \lambda)\right]$$

$$(j, k = 1, 2, \dots, 2n),$$

where x_0 is any fixed point of the interval (a, b). The matrix function

$$M(\lambda) = \left\| m_{jk}(\lambda) \right\|_1^{2n}$$

will be called the characteristic matrix of the generalized resolvent R_λ.**

Theorem 1. *The characteristic matrix $M(\lambda)$ of any generalized resolvent R_λ of the operator L is a regular function of the parameter λ in the upper and lower half-planes, and*

$$M(\overline{\lambda}) = M^*(\lambda).$$

For any λ in the upper half-plane the matrix

$$\operatorname{Im} M(\lambda) = \frac{1}{2i}\,[M(\lambda) - M^*(\lambda)]$$

is Hermitian nonnegative, $\operatorname{Im} M(\lambda) \geq 0$.

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* The basic definitions and facts to which we refer here are set out, for example, in ([1,2]).

** In ([4]) the matrix $M(\lambda)$ is defined directly by formula (9). We note that in that formula the factor 1/2 is omitted at the matrix $Q_0(\lambda)$.

For any complex λ, let us consider the system of functions

$$y_1(x;\lambda),\ y_2(x;\lambda), \dots,\ y_{2n}(x;\lambda), \tag{1}$$

which are solutions of the equation

$$l[y] - \lambda y = 0$$

and satisfy the conditions

$$y_k^{[j-1]}(x_0;\lambda) = \delta_{jk} \qquad (j,k = 1,2,\dots,2n),$$

where the expression on the left denotes the quasiderivative of order $j-1$.

By $y(x;\lambda)$ we shall denote the column matrix composed of the functions (1), and by $y'(x;\lambda)$ the corresponding transposed matrix.

Let E_t $(-\infty < t < +\infty)$ be an arbitrary spectral function of the operator L (in general nonorthogonal), and let R_λ be the corresponding generalized resolvent. Applying the Stieltjes inversion formula to the generalized resolvent R_λ, we obtain the formula for all spectral functions of the operator L.

Theorem 2. For any real α and β, the operator

$$E_{\alpha,\beta} = \frac{E_\beta + E_{\beta+0}}{2} - \frac{E_\alpha + E_{\alpha+0}}{2}$$

is integral. Its kernel is determined by the formula

$$K(x,s;\alpha,\beta)=\int_\alpha^\beta y'(x;\lambda)\,dT(\lambda)\,y(s;\lambda),$$

where

$$T(\lambda)=\frac{1}{\pi}\lim_{\tau\to+0}\int_0^\lambda \operatorname{Im} M(\sigma+i\tau)\,d\sigma.$$

Moreover,

$$\int_a^b |K(x,s;\alpha,\beta)|^2\,ds<\infty;\qquad \int_a^b |K(x,s;\alpha,\beta)|^2\,dx<\infty$$

$$(a<x,\ \ s<b).$$

The following assertion is closely connected with Theorem 2.

Theorem 3. For any function $f(x)\in L^2(a,b)$ the expansion

$$f(x)=\int_{-\infty}^{+\infty} y'(x;\lambda)\,dT(\lambda)\,\eta(f;\lambda), \tag{2}$$

holds, where

$$\eta(f;\lambda)=\int_a^b f(s)y(s;\lambda)\,ds. \tag{3}$$

Moreover, the integrals in the right-hand sides of formulas (2), (3) converge correspond–

respectively in the sense of the metrics of the spaces $L^2(a,b)$ and $L^2_T(-\infty,+\infty)$, and the equality

$$\int_a^b |f(x)|^2\,dx=\int_{-\infty}^{+\infty}\eta^*(f;\lambda)\,dT(\lambda)\,\eta(f;\lambda)$$

holds.

3. Suppose now that both endpoints of the interval (a,b) are regular. Let $A(\lambda)$ and $B(\lambda)$ be square matrices of order $2n$, depending on the complex parameter λ and satisfying the following conditions:

a) the rank of the rectangular matrix $\|A(\lambda)\ B(\lambda)\|$ is equal to $2n$ for every λ;

b) $A(\lambda)$ and $B(\lambda)$ are entire matrix functions of the parameter λ;

c) for every non-real λ,

$$\frac{1}{\lambda-\bar{\lambda}}\left[B(\lambda)JB^{*}(\lambda)-A(\lambda)JA^{*}(\lambda)\right]\geq 0,$$

where the skew-symmetric matrix J of order $2n$ has the form

$$J=\begin{pmatrix} & & & -1 \\ & & -1 & \\ & \iddots & & \\ -1 & & & \\ 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

and all unmarked positions contain zeros.

Every function $y(x)$ for which $l[y]$ has meaning possesses quasi-derivatives $y^{[j]}(x)$ ($j=0,1,\dots,2n-1$) that are absolutely continuous on the interval $[a,b]$. By $\hat{y}(x)$ we denote the one-column matrix function composed of these quasi-derivatives:

$$\hat{y}(x)=(y^{[0]}(x),y^{[1]}(x),\dots,y^{[2n-1]}(x)).$$

Consider the following boundary-value problem:

$$l[y]-\lambda y=0, \tag{4}$$

$$A(\lambda)\hat{y}(a)+B(\lambda)\hat{y}(b)=0^{*}. \tag{5}$$

To the boundary condition (5) there corresponds a certain generalized resolvent R_{λ} of the operator L; for any function $f(x)\in L^{2}(a,b)$, $R_{\lambda}f$, for every non-real λ, is defined as the solution of the equation

$$l[y]-\lambda y=f,$$

satisfying the boundary condition (5) (cf. (4)). Owing to this circumstance, propositions on generalized resolvents and the corresponding spectral functions prove useful in the study of the boundary-value problem (4), (5).

Theorem 4. All eigenvalues of the boundary-value problem (4), (5) are real, and their set is countable and coincides with the set of poles of the meromorphic matrix function

$$M(\lambda) = \frac{1}{2}[A(\lambda)Y(a;\lambda) + B(\lambda)Y(b;\lambda)]^{-1}\times$$
$$\times[A(\lambda)Y(a;\lambda) - B(\lambda)Y(b;\lambda)]J^{-1},$$

* Boundary-value problems of this type occur in applications (see, for example, (^{3}), p. 144).

where $Y(x;\lambda)$ is the matrix of quasi-derivatives corresponding to the system (1), i.e.

$$Y(x;\lambda) = \| \,\hat{y}_1(x;\lambda);\ \hat{y}_2(x;\lambda) \ldots \hat{y}_{2n}(x;\lambda)\, \| \,.$$

All these poles are simple.

Let

$$\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots$$

be the sequence of all eigenvalues of the boundary-value problem (4), (5).

Denote by T_k the residue of the matrix function $-M(\lambda)$ at the point λ_k. Put

$$H_k(x;s) = y'(x;\lambda_k)T_k y(s;\lambda_k) \qquad (a \le x, s \le b; \quad k = 1, 2, \ldots)$$

and introduce into consideration the integral operators Q_k in $L^2(a,b)$:

$$Q_k f = \int_a^b H_k(x;s) f(s)\, ds \qquad (k = 1, 2, \ldots).$$

For any k, the operator Q_k is, obviously, nonnegative.

Denote by $\mathfrak{M}_k$ the subspace of eigenfunctions of the boundary-value problem (4), (5) corresponding to the eigenvalue λ_k $(k = 1, 2, \ldots)$.

Theorem 5. For any k $(k = 1, 2, \ldots)$,

$$Q_k L^2(a,b) = Q_k \mathfrak{M}_k = \mathfrak{M}_k.$$

Theorem 6. Every function $f(x)$ from $L^2(a,b)$ can be expanded in a series, convergent in the metric of this space, in the eigenfunctions of the boundary-value problem (4), (5):

$$f = \sum_{k=1}^{\infty} Q_k f.$$

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Note: Figure translations are in progress. See original paper for figures.

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