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Abstract

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SOME QUESTIONS OF HARMONIC ANALYSIS IN HOMOGENEOUS CONES

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The present work is devoted to the generalization and refinement of a formula of Siegel—Maass—Selberg, which is of great importance in the theory of Dirichlet series.

Let us agree to associate with each real symmetric matrix $Y = (y_{ks})$ of order p the point of $n = \frac{p(p+1)}{2}$-dimensional Euclidean space with coordinates y_{ks}, $k \leqslant s$. The totality of positive matrices, i.e. matrices with positive eigenvalues, forms a certain convex homogeneous* cone V.

Siegel established the formula (^{1})

$$\int_V e^{-\sigma(YT)}|Y|^{s-(p+1)/2}\,dY = \pi^{p(p-1)/4}\Gamma(s)\dots\Gamma\left(s-\frac{p-1}{2}\right)|T|^{-s},$$

$$dY = \prod_{k\leqslant s} dy_{ks} \tag{1}$$

($\sigma(A)$ is the trace of the matrix A; $|A|$ is the determinant of the matrix A).

Maass found a generalization of Siegel' s formula (^{2})

$$\int e^{-\sigma(YT)}u(Y)|Y|^{s-(p+1)/2}\,dY =$$

$$= \pi^{p(p-1)/4}\Gamma(s-\gamma_1)\dots\Gamma(s-\gamma_p)\,u(T^{-1})(T)^{-s}, \tag{2}$$

where $u(Y)$ is a bounded solution, introduced by Selberg (^{2,4}), of the system of partial differential equations

$$\left(\sigma\left(Y\frac{\partial}{\partial Y}\right)^k + \lambda_k\right)u(Y) = 0, \qquad k = 1, 2, \dots, p, \tag{3}$$

where

$$\partial/\partial Y = (e_{\mu\nu}\partial/\partial y_{\mu\nu}); \qquad e_{\mu\mu} = 1; \quad e_{\mu\nu} = 1/2\ (\mu \neq \nu).$$

Maass' s method is based on rather complicated calculations and does not make it possible to find the relation between $\gamma_1, \dots, \gamma_p$ and $\lambda_1, \dots, \lambda_p$.

In the present note a simple proof of formula (2) and of some of its generalizations is given, making it possible to find effectively $\gamma_1, \gamma_2, \dots, \gamma_p$.

The differential operators $\sigma(Y\partial/\partial Y)^k$ commute with the operators of motion $T_R f(Y) = f(R'YR)$; following the terminology proposed by I. M. Gel' fand (^{6}), we shall call them **harmonic operators**. The totality of all functions satisfying the system of equations (3) with fixed $\lambda_1, \dots, \lambda_p$ forms a space in which there acts a certain representation of the group of all nondegenerate real matrices of order p. This representation is, generally speaking, irreducible. Using this fact and a theorem of Bochner, it is easy to derive (2).

The integral operator

$$Kf = \int_{\dot V} e^{-\sigma(YT^{-1})} f(Y) |Y|^{-(p+1)/2}\, dY \tag{4}$$

commutes with the operators T_R, and, consequently, any function from an irreducible representation is an eigenfunction for it. It is easy to verify that, together with $u(Y)$, the function $u(Y)|Y|^s$ is also an eigenfunction of the integral operator (4), with eigenvalue depending on s. Therefore

$$\int_{\dot V} e^{-\sigma(YT)} u(Y) |Y|^{s-(p+1)/2}\, dY = G(s) u(T^{-1}) |T|^{-s}. \tag{5}$$

From Bochner' s theorem (^{3}) it follows that

$$G(s) = A^s \prod_{k=1}^{p} \Gamma(s - \gamma_k).$$

Derivation of formula (2), making it possible to establish the connection between $\gamma_0, \dots, \gamma_p$ and $\lambda_1, \dots, \lambda_p$. It is known that among the solutions of (3) there is always a function of the form

$$e_{\alpha_1, \dots, \alpha_p}(Y) = \Delta_1^{\alpha_1} \Delta_2^{\alpha_2} \cdots \Delta_p^{\alpha_p}, \tag{6}$$

* A cone is called **homogeneous** if the group of affine transformations carrying it onto itself is transitive on it.

where $\Delta_1 = y_{11}$, $\Delta_2 = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix}$, ..., is the sequence of principal minors of the matrix Y. The eigenvalues $\lambda_1, \dots, \lambda_p$ are symmetric functions of

$$\beta_1 = \alpha_1 + \dots + \alpha_p, \quad \beta_2 = \alpha_2 + \dots + \alpha_p - \frac{1}{2}, \dots, \quad \beta_p = \alpha_p - \frac{p-1}{2}$$

($^{5-7}$).

Using this circumstance and formula (14) from the work of Maass (^{2}), one can write an explicit expression for $\lambda_1, \dots, \lambda_p$ in terms of $\beta_1, \dots, \beta_p$. We do not give it, since we regard the most natural parameters defining the representation to be precisely $\beta_1, \dots, \beta_p$.

We shall prove that the following formula holds:

$$\int_{\dot{V}} e^{-\sigma(TY)} e_{\alpha_1,\dots,\alpha_p}(Y) |Y|^{-(p+1)/2}\, dY = G_{\alpha_1,\dots,\alpha_p} e_{\alpha_1,\dots,\alpha_p}(T^{-1}),$$

$$G_{\alpha_1,\dots,\alpha_p} = \pi^{p(p-1)/4} \Gamma(\beta_1) \cdots \Gamma(\beta_p). \tag{7}$$

Denote by K the group of all matrices Q all of whose elements below the main diagonal are equal to zero, while the elements on the main diagonal are positive. It is clear that

$$e_{\alpha_1,\dots,\alpha_p}(Q'YQ) = q_1^{\alpha_1+\dots+\alpha_p} q_2^{\alpha_2+\dots+\alpha_p} \cdots q_p^{\alpha_p} e_{\alpha_1,\dots,\alpha_p}(Y),$$

where $q_1, q_2, \dots, q_p$ are the diagonal elements of Q.

As is known, any positive-definite matrix T can be represented in the form $Q^{-1}Q'^{-1}$, $Q \in K$. Making the change of variables $Y = Q'\widetilde{Y}Q$, where $Q^{-1}Q'^{-1} = T$, we obtain that it is sufficient to prove formula (7) for $T = E$. Represent the matrix Y in the form

$$Y = \begin{pmatrix} Y_1 & q \\ q' & y \end{pmatrix},$$

where Y_1 is a matrix of order $p-1$; q is a $(p-1)$-dimensional vector. Then

$$|Y| = |Y_1|(y - q'Y_1^{-1}q), \qquad \sigma(Y) = \sigma(Y_1) + y.$$

Thus, $Y > 0$ if and only if $Y_1 > 0$ and $y - q'Y_1^{-1}q > 0$. It is easy to derive the recurrence formula

$$\int_V e^{-\sigma(Y)} f(Y_1) |Y_1|^{s-(p+1)/2}\, dY =$$

$$= \pi^{(p-1)/2}\Gamma\left(s - \frac{p-1}{2}\right)\int_{V_1} e^{-\sigma(Y_1)} f(Y_1)|Y_1|^{s-p/2}\, dY_1; \tag{8}$$

$f(Y_1)$ is any function for which the integral on the right-hand side converges absolutely.

Formula (7) easily follows from (8) by induction.

In note (9), four series of homogeneous convex cones in Euclidean space are listed. One of these series is the cones of positive symmetric matrices; for each of the remaining three series there exist formulas analogous to (7).

I. Consider the set V of all positive Hermitian matrices $Y = (y_{ks})$ of order p. Obviously, V is a cone in p^2-dimensional Euclidean space. The invariant volume element is defined by the formula

$$dv = |Y|^{-p} \prod_{k \le s} du_{ks} \prod_{k<s} dv_{ks},$$

where $u_{ks} = \operatorname{Re} y_{ks}$, $v_{ks} = \operatorname{Im} y_{ks}$. Put $e_{\alpha_1,\dots,\alpha_p}(Y) = \Delta_1^{\alpha_1} \cdots \Delta_p^{\alpha_p}$, where $\Delta_1, \Delta_2, \dots$ is the sequence of principal minors of the matrix Y. The formula holds

$$\int_V e^{-\sigma(TY)} e_{\alpha_1,\dots,\alpha_p}(Y)\, dv = G_{\alpha_1,\dots,\alpha_p} e_{\alpha_1,\dots,\alpha_p}(T^{-1}), \tag{9}$$

$$G_{\alpha_1,\dots,\alpha_p} = \pi^{p(p-1)/2}\Gamma(\alpha_1 + \dots + \alpha_p)\Gamma(\alpha_2 + \dots + \alpha_p - 1) \cdots \Gamma(\alpha_p - p + 1).$$

II. V is the set of all Hermitian matrices Y of order $2p$ for which the relation

$$YJ = J\bar{Y}, \qquad J = \begin{pmatrix} j & & 0 \\ & \ddots & \\ 0 & & j \end{pmatrix}, \qquad i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{10}$$

It is easy to verify that V is the set of all positive-definite matrices of the algebra of quaternions.

Put $Y = (y_{ks})$, $k, s = 1, 2, \dots, p$, where y_{ks} are matrices of the second order. Relation (10) can be rewritten in the following form:

$$y'_{ks} = \bar{y}_{sk}, \qquad y_{ks} j = j \bar{y}_{ks}.$$

Hence,

$$y_{kk} = \begin{pmatrix} u_{kk} & 0 \\ 0 & u_{kk} \end{pmatrix}, \qquad u_{kk} > 0;$$

$$y_{ks} = \begin{pmatrix} u_{ks} & v_{ks} \\ -\bar{v}_{ks} & \bar{u}_{ks} \end{pmatrix} \quad (k < s), \qquad u_{ks} = a_{ks} + ib_{ks}, \quad v_{ks} = c_{ks} + id_{ks},$$

where $u_{kk}, a_{ks}, b_{ks}, c_{ks}, d_{ks}$ are real numbers.

The invariant volume element is defined by the formula

$$dv = |Y|^{1/2-p} \prod_{1}^{p} du_{kk} \prod_{k<s} da_{ks} \prod_{k<s} db_{ks} \prod_{k<s} dc_{ks} \prod_{k<s} dd_{ks}.$$

Put $e_{\alpha_1,\dots,\alpha_p}(Y) = \Delta_1^{\alpha_1/2} \cdots \Delta_p^{\alpha_p/2}$, where $\Delta_1 = |y_{11}|$, $\Delta_2 = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix}, \dots$

The formula holds

$$\int_V e^{-\frac{1}{2}\sigma(YT)} e_{\alpha_1,\dots,\alpha_p}(Y)\, dv = G_{\alpha_1,\dots,\alpha_p} e_{\alpha_1,\dots,\alpha_p}(T^{-1}), \tag{11}$$

$$G_{\alpha_1,\dots,\alpha_p} = \pi^{p(p-1)} \Gamma(\alpha_1 + \dots + \alpha_p) \Gamma(\alpha_2 + \dots + \alpha_p - 2) \cdots \Gamma(\alpha_p - 2p + 2).$$

III. V is the set of all points $y = (y_1, \dots, y_n)$ of n-dimensional space for which $y_1 y_2 - y_3^2 - \dots - y_n^2 > 0$, $y_1 > 0$. The invariant volume is defined by the formula $dv = (\lambda(y))^{-n/2} dy_1 \cdots dy_n$, $\lambda(y) = y_1 y_2 - y_3^2 - \dots - y_n^2$. Put $e_{\alpha,\beta}(y) = y_1^{\alpha} (\lambda(y))^{\beta}$.

The formula holds

$$\int_V e^{-(y,t)} e_{\alpha,\beta}(y)\, dv = G_{\alpha,\beta} e_{\alpha,\beta}(t^{-1}), \qquad G_{\alpha,\beta} = \pi^{n/2-1} \Gamma(\alpha + \beta) \Gamma\left(\beta - \frac{n}{2} + 1\right), \tag{12}$$

$$t^{-1} = \left(\frac{t_2}{\lambda(t)}, \frac{t_1}{\lambda(t)}, -\frac{t_3}{\lambda(t)}, \dots, -\frac{t_n}{\lambda(t)} \right), \qquad (y, t) = y_1 t_1 + y_2 t_2 + 2 \sum_{3}^{n} y_{kt} k.$$

We indicate one application of the formulas derived. Let V be a cone in Euclidean space E_n which is a product of the cones listed above; (y, t) is the scalar product. We shall call the **mutual cone** the set of all vectors $t \in E_n$ for which $(y, t) > 0$ for all $y \in V$. It is easy to see that, under a suitable choice of scalar product, the mutual cone coincides with V. Let $y \to y^{\varphi}$ be a mapping of V onto itself possessing the following properties: for any two vectors $y, t \in V$ and any affine transformation L of the cone V into itself, $(y, t^\varphi) = (Ly, (Lt)^\varphi)$. Denote by H the set of all points of n-dimensional complex space of the form $y + ix$, $y \in V$, x arbitrary. The mapping $y \to y^\varphi$ extends to an analytic automorphism of H onto itself. Let $\lambda(y)$ be a polynomial in y such that $\lambda(Ly) = |L|\lambda(y)$ for any linear transformation L of the cone V into itself.

We shall call an eigenfunction of all harmonic operators on V a harmonic function. Let $u(y)$ be a harmonic function for which

$$\int_V e^{-(y,t)}|u(y)|\lambda^{-1}(y)\,dy < \infty \qquad \text{for all } t \in V. \tag{13}$$

From formulas (7), (9), (11), and (12) it follows that

$$Gu(t^\varphi) = \int_V e^{-(y,t)}u(y)\lambda^{-1}(y)\,dy.$$

It is easy to see that the function $v(t) = u(t^\varphi)$ extends analytically to H. Let Γ be a lattice in n-dimensional space, and Γ' the reciprocal lattice. With the aid of the Poisson formula one easily obtains the identity

$$\sum_{r\in\Gamma} v(t + 2\pi i r) = \frac{1}{G|\Gamma|}\sum_{\substack{\rho\in\Gamma' \\ \rho\in V}} u(\rho)e^{-(\rho,t)}\lambda^{-1}(\rho); \tag{14}$$

$|\Gamma|$ is the volume of the fundamental parallelepiped of the lattice Γ. Formula (14) holds under hypotheses ensuring convergence of the left- and right-hand sides.

In the special case when V is the set of all positive-definite symmetric matrices Y, $u(Y) = |Y|^s$, this formula was proved by Siegel (^{1}).

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CITED LITERATURE

1. C. Siegel, Ann. Math., **36**, 527 (1935).
2. H. Maass, J. Ind. Math. Soc., **19**, No. 1, 1 (1955).

3. S. Bochner, Proc. Nat. Acad. Sci. USA, **42**, No. 2, 86 (1956).
4. A. Selberg, Intern. Coll. on Zeta-Functions, 1956.
5. I. M. Gelfand, A. M. Naimark, Tr. Matem. inst. AN SSSR im. V. A. Steklova, **26** (1950).
6. I. M. Gelfand, DAN, **70**, No. 1 (1950).
7. F. A. Berezin, DAN, **107**, No. 1 (1956).
8. I. I. Pyatetskii-Shapiro, DAN, **113**, No. 5 (1957).

Note: Figure translations are in progress. See original paper for figures.