ON A-CAUCHY INTEGRALS FOR CONTOURS

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Abstract

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MATHEMATICS

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Let l be a rectifiable closed contour on which an L-integrable function $f(\zeta)$ is given. Then the integral

$$\frac{1}{2\pi i}\,(L)\int_l \frac{f(\zeta)}{\zeta - z}\,d\zeta = F(z), \tag{1}$$

where the point z lies inside the contour l, and the integration is taken counterclockwise, is called an integral of Cauchy type over the closed contour l.

If the contour l is sufficiently smooth, then it is known that the analytic function $F(z)$, as z approaches the contour l, has nontangential boundary values from inside this contour for almost all points of the contour l ([2]). Let $F_i(\zeta)$ be the boundary values of the function $F(z)$ from inside the contour l, where $\zeta \in l$. The function $F_i(\zeta)$ need not coincide with the function $f(\zeta)$. Moreover, the function $F_i(\zeta)$, generally speaking, is not integrable on l either in the sense of Lebesgue or in the sense of Denjoy.

The present note is devoted to the question of the possibility of representing the analytic function $F(z)$ in terms of $F_i(\zeta)$ by a formula analogous to (1), in which the function $F_i(\zeta)$ would stand in place of $f(\zeta)$.

A narrower problem was solved by us in the notes ([5,7]).

It follows from all that has been said that, in solving the question posed, the use of integration in the sense of Lebesgue or in the sense of Denjoy cannot lead to the goal. Therefore we introduce a certain new integration.

Definition. Let a smooth contour l of length l_0 be given in the complex ζ-plane, with initial point ζ_0 and terminal point ζ_0', and let the equation of the contour be

$$\zeta = \tau(s) = \tau_1(s) + i\tau_2(s),$$

where s is the length of the arc of the curve l from the point ζ_0 to the point ζ $(\zeta_0 = \tau(0),\ \zeta_0' = \tau(l_0))$. Then a function

$$f(\zeta) = f_1(s) + if_2(s),$$

defined on l, is called A-integrable on l if each of the functions

$$\varphi_1(s) = [f_1(s)\tau_1'(s) - f_2(s)\tau_2'(s)],$$

$$\varphi_2(s) = [f_2(s)\tau_1'(s) + f_1(s)\tau_2'(s)]$$

is A-integrable on $0 \le s \le l_0$. The complex number

$$I = (A)\int_0^{l_0} \varphi_1(s)\,ds + i(A)\int_0^{l_0} \varphi_2(s)\,ds$$

will, by definition, be called the A-integral of the function $f(\zeta)$ along the curve l:

$$(A)\int_l f(\zeta)\,d\zeta = I. \tag{2}$$

(For the definition and properties of the A-integral on an interval, see ([1,3,4,6]).)

If the contour l is the unit circle, then $\zeta = e^{is}$ $(0 \le s \le 2\pi)$, and for this case formula (2) gives

$$(A)\int_{|\zeta|=1} f(\zeta)\,d\zeta = -(A)\int_0^{2\pi} [f_1(s)\sin s + f_2(s)\cos s]\,ds + i(A)\int_0^{2\pi} [f_1(s)\cos s - f_2(s)\sin s]\,ds, \tag{3}$$

i.e., we obtain the definition of the A-integral over a circle, which was introduced by us earlier ([8]).

For A-integrals of real functions defined on an interval, the formula for change of variables is valid. Namely, the following theorem holds.

Theorem 1. *If $f(x) \in A(a,b)$ and the function $x = \varphi(y)$ is such that it maps some interval $[c,d]$ onto $[a,b]$ and, moreover, $0 < B_1 \le |\varphi'(y)| \le B_2 < \infty$, then*

$$(A)\int_a^b f(x)\,dx = (A)\int_c^d f(\varphi(y))\varphi'(y)\,dy, \tag{4}$$

where B_1 and B_2 are certain constants.

Now we can formulate the main result.

Theorem 2. *Let l be a closed contour bounding a domain G, and let the equation of the contour have the form $\zeta = \tau(s) = x(s) + iy(s)$, where the functions $x(s)$ and $y(s)$ satisfy the condition*

$$|x'(s_2) - x'(s_1)| \le k|s_2 - s_1|^{\alpha}, \qquad |y'(s_2) - y'(s_1)| \le k|s_2 - s_1|^{\alpha} \tag{5}$$

for all s_1 and s_2, with certain constants $k > 0$ and $\alpha > 0$. Then, if the analytic function $F(z)$ is representable in the domain G by an L-integral of Cauchy type, i.e. if

$$F(z) = \frac{1}{2\pi i}(L)\int_l \frac{f(\zeta)}{\zeta - z}\,d\zeta \qquad (z \in G,\ f(\zeta) \in L(l)),$$

then

$$F(z) = \frac{1}{2\pi i}(A)\int_l \frac{F_i(\zeta)}{\zeta - z}\,d\zeta, \tag{6}$$

where $F_i(\zeta)$ are the limiting values of the function $F(z)$ from inside the domain G.

In other words, under the conditions (5) imposed on the contour l:
*Every L-integral of Cauchy type is an A-integral of Cauchy type.**

From this theorem we may conclude that the limiting values of an integral of Cauchy type for a closed contour are A-integrable functions on this contour.

Theorem 3. *Let an analytic function $F(z)$ be representable as an L-integral of Cauchy type in a domain G bounded by a contour l, and let $\beta(z)$ be a bounded analytic function in the domain G. Then*

$$(A)\int_l F_i(\zeta)\beta(\zeta)\,d\zeta = 0. \tag{7}$$

* A special case of this theorem (the l-circle) was treated by us in note ([8]).

Corollary. If for $\beta(z)$ one takes $\beta(z) = z^n$ $(n = 0, 1, ...)$, then it follows from Theorem 3 that

$$(A)\int_l F_i(\zeta)\zeta^n\,d\zeta = 0 \qquad (n = 0, 1, ...), \tag{8}$$

i.e., all moments of the function $F_i(\zeta)$ over the contour l are equal to zero.

Thus, the boundary values of analytic functions $F(z)$ that are representable by an L-integral of Cauchy type for closed contours possess, from the standpoint of A-integration, a number of properties entirely analogous to the properties of analytic functions of the class H_1 or E_1.

Remark 1. One can likewise consider L-integrals of Cauchy type for open contours. In this case one can also obtain a number of results concerning the properties of the boundary values of analytic functions representable by an L-Cauchy integral for a certain open contour.

Remark 2. The conditions (5) imposed on the contour l can, generally speaking, be weakened still further. Since the weakened conditions are more cumbersome than conditions (5), we do not give them.

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Note: Figure translations are in progress. See original paper for figures.