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Abstract

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MECHANICS

L. I. RUBINSHTEIN

ON THE SOLUTION OF N. N. VERIGIN' S PROBLEM

(Presented by Academician S. L. Sobolev, 22 IX 1956)

N. N. Verigin ⁽¹⁾ considered a problem that is related to the well-known Stefan problem ⁽²⁾ in the same way as ordinary two-layer heat problems are related to ordinary one-layer problems. N. N. Verigin' s example is one of the few (among problems of this kind) that admit a self-similar solution. In the general formulation, special methods must be developed in order to solve Verigin' s problem. This is done below for Verigin' s problem in the following formulation.

It is required to determine the functions p_1, p_2 , and y , satisfying the conditions:

$$\frac{\partial^2 p_1}{\partial \xi^2} = \frac{\partial p_1}{\partial \tau} \quad \text{for } -\infty < \xi < y(\tau), \tau > 0; \tag{1}$$

$$a^2 \frac{\partial^2 p_2}{\partial \xi^2} = \frac{\partial p_2}{\partial \tau} \quad \text{for } y(\tau) < \xi < 0, \tau > 0; \tag{2}$$

$$p_i|_{\tau=0} = \varphi_i(\xi) \quad (i = 1, 2); \quad \left. \frac{\partial p_2}{\partial \xi} \right|_{\xi=0} = \psi(\tau); \tag{3}$$

$$p_1|_{\xi=y(\tau)-0} = p_2|_{\xi=y(\tau)+0}; \quad \lambda \left. \frac{\partial p_1}{\partial \xi} \right|_{\xi=y(\tau)-0} = \left. \frac{\partial p_2}{\partial \xi} \right|_{\xi=y(\tau)+0} \tag{4}$$

$$\frac{dy}{d\tau} = - \left. \frac{\partial p_1}{\partial \xi} \right|_{\xi=y(\tau)}; \quad y(0) = -1. \tag{5}$$

It is assumed that $\varphi_i(\xi)$ is three times and $\psi(\tau)$ twice continuously differentiable everywhere in their domains of definition, and moreover

$$\left| \frac{d^m \varphi_i}{d\xi^m} \right|, \left| \frac{d^k \psi}{d\tau^k} \right| < M \quad (m = 0, 1, 2, 3; k = 0, 1, 2) \tag{6}$$

\end{equation}

and, in addition,

$$\varphi_2(-1) = \varphi_1(-1); \quad a\dot{\varphi}_2(-1) + \dot{\varphi}_1(-1) \neq 0; \quad \dot{\varphi}_2(0) = \psi(0). \quad (7)$$

Put

$$\left. \frac{\partial p_1}{\partial \xi} \right|_{\xi=y(\tau)} = v(\tau); \quad p_i|_{\xi=y(\tau)} = f(\tau); \quad p_2|_{\xi=0} = F(\tau) \quad (8)$$

and denote by $g_i(\xi, \xi_1, \tau - \tau_1)$ and $G_i(\xi, \xi_1, \tau - \tau_1)$ the Green's functions of the first and second boundary-value problems for equations (1)–(5) for the half-line $\xi < 0$. The following equalities must hold:

$$\begin{aligned} p_2(\xi, \tau) &= \int_{-1}^0 \varphi_2(\xi_1) G_2(\xi, \xi_1, \tau) d\xi_1 + a^2 \int_0^\tau f(\tau_1) \frac{\partial}{\partial \xi_1} G_2(\xi, y(\tau_1), \tau - \tau_1) d\tau_1 \\ &\quad - \int_0^\tau [a^2 v(\tau_1) + \dot{y}(\tau_1) f(\tau_1)] G_2(\xi, y(\tau_1), \tau - \tau_1) d\tau_1 \\ &\quad + a^2 \int_0^\tau \psi(\tau_1) G_2(\xi, 0, \tau - \tau_1) d\tau_1 = \sum_{i=2}^4 J_i^0(\xi, \tau). \end{aligned} \quad (9)$$

In the limit as $\xi \rightarrow 0$ we find:

$$F(\tau) = \sum_{i=1}^4 J_i^0(0, \tau). \quad (10)$$

Let us now make in (1)–(5) the change of variables, putting

$$x = \xi - y(\tau); \quad t = \tau; \quad z(t) = -y(\tau); \quad p_i^*(x, t) = p_i(\xi, \tau);$$

$$q_i(x, t) = \frac{\partial}{\partial x} p_i^*(x, t); \quad f_i(x) = \varphi_i(x - 1).$$

We obtain

$$\frac{\partial^2 p_1^*}{\partial x^2} = \frac{\partial p_1^*}{\partial t} + \dot{z}(t) q_1(x, t) \quad \text{for } -\infty < x < 0, t > 0; \quad (12)$$

$$a^2 \frac{\partial^2 p_2^*}{\partial x^2} = \frac{\partial p_2^*}{\partial t} + \dot{z}(t) q_2(x, t) \quad \text{for } 0 < x < z(t), t > 0; \quad (13)$$

$$p_i^*|_{t=0} = f_i(x); \quad \left. \frac{\partial}{\partial x} p_2^* \right|_{x=z(t)} = \psi(t); \quad (14)$$

$$p_1^*|_{x=0} = p_2^*|_{x=0}; \quad \lambda \frac{\partial}{\partial x} p_1^*|_{x=0} = \frac{\partial}{\partial x} p_2^*|_{x=0}; \quad (15)$$

$$\frac{dz}{dt} = -\frac{\partial p_1^*}{\partial x} \Big|_{x=0}; \quad z(0) = 1. \quad (16)$$

Subjecting the equalities (12) and (13) to Green' s transformation and taking into account (8) and (14), we find:

$$p_1^* = \int_{-\infty}^0 f_1(\xi) G_1(x, \xi, t) d\xi + \int_0^t v(\tau) G_1(x, 0, t-\tau) d\tau - \int_0^t \dot{z}(\tau) d\tau \int_{-\infty}^0 q_1(\xi, \tau) G_1(x, \xi, t-\tau) d\xi = \sum_1^3 J_i^{(1)}(x, t); \quad (17)$$

$$\begin{aligned} p_2^* &= \int_0^1 f_2(\xi) G_2(x, \xi, t) d\xi - \lambda a^2 \int_0^t v(\tau) G_2(x, 0, t-\tau) d\tau \\ &\quad - \int_0^t F(\tau) \left[a^2 \frac{\partial}{\partial \xi} - \dot{z}(\tau) \right] G_2(x, z(\tau), t-\tau) d\tau + a^2 \int_0^t \psi(\tau) G_2(x, z(\tau), t-\tau) d\tau \\ &\quad - \int_0^t \dot{z}(\tau) d\tau \int_0^{z(\tau)} q_2(\xi, \tau) G_2(x, \xi, t-\tau) d\xi = \sum_{i=1}^5 J_i^{(2)}(x, t). \end{aligned} \quad (18)$$

Passing in (17), (18) to the limit as $x \rightarrow 0$ and taking into account the first of the conditions (15), we arrive at an Abel-type integral equation with respect to $v(t)$. Inverting this equation gives

$$v(t) = \frac{1}{1 + \lambda a} \sum_{i=1}^6 I_i(t), \quad (19)$$

where

$$\begin{aligned} I_1 &= \int_{-\infty}^0 \dot{f}_1(\xi) G_1(0, \xi, t) d\xi; & I_2 &= a \int_0^1 \dot{f}_2(\xi) G_2(0, \xi, t) d\xi; \\ I_3 &= a \int_0^t \dot{F}(\tau) G_2(0, z(\tau), t-\tau) d\tau; & I_4 &= a^3 \int_0^t \psi(\tau) \frac{\partial}{\partial \xi} G_2(0, z(\tau), t-\tau) d\tau; \\ I_5 &= \int_0^t \dot{z}(\tau) d\tau \int_{-\infty}^0 q_1(\xi, \tau) \frac{\partial}{\partial \xi} G_1(0, \xi, t-\tau) d\xi; \end{aligned} \quad (20)$$

$$I_6 = a \int_0^t \dot{z}(\tau) d\tau \int_0^{z(\tau)} q_2(\xi, \tau) \frac{\partial}{\partial \xi} G_2(0, \xi, t - \tau) d\xi. \quad (20a)$$

Introducing, further, $t, z,$ and f_2 in (10) instead of $\tau, y,$ and $\varphi_2,$ we readily find:

$$\begin{aligned} \dot{F}(t) = & a^2 \left[\dot{f}_2(0) G_2(0, -1, t) - f_2(0) \frac{\partial}{\partial \xi} G_2(0, -1, t) \right] + \\ & + a^2 \int_{-1}^0 \dot{f}_2(\xi + 1) G_2(0, \xi, t) d\xi + a^2 \int_0^t f(\tau) \frac{\partial^2}{\partial \xi \partial t} G_2(0, -z(\tau), t - \tau) d\tau - \\ & - \int_0^t [a^2 v(\tau) - \dot{z}(\tau) f(\tau)] \frac{\partial}{\partial t} G_2(0, -z(\tau), t - \tau) d\tau + \\ & + a^2 \int_0^t \dot{\psi}(\tau) G_2(0, 0, t - \tau) d\tau \equiv R_0(t) + \sum_{j=1}^4 J_{0j}. \end{aligned} \quad (21)$$

Finally, differentiating (17) and (18) with respect to $x,$ after simple transformations taking (6)–(7) and (11) into account, we find that

$$\begin{aligned} q_1(x, t) = & \int_{-\infty}^0 \dot{f}_1(\xi) g_1(x, \xi, t) d\xi - \int_0^t v(\tau) \frac{\partial}{\partial \xi} g_1(x, 0, t - \tau) d\tau + \\ & + \int_0^t \dot{z}(\tau) d\tau \int_{-\infty}^0 q_1(\xi, \tau) \frac{\partial}{\partial \xi} g_1(x, \xi, t - \tau) d\xi \equiv \sum_{j=1}^3 J_{1j}; \end{aligned} \quad (22)$$

$$\begin{aligned} q_2(x, t) = & \int_0^1 \dot{f}_2(\xi) g_2(x, \xi, t) d\xi + \lambda a^2 \int_0^t v(\tau) \frac{\partial}{\partial \xi} g_2(x, 0, t - \tau) d\tau + \\ & + \int_0^t \dot{F}(\tau) g_2(x, z(\tau), t - \tau) d\tau - a^2 \int_0^t \dot{\psi}(\tau) \frac{\partial}{\partial \xi} g_2(x, z(\tau), t - \tau) d\tau + \\ & + \int_0^t \dot{z}(\tau) d\tau \int_0^{z(\tau)} q_2(\xi, \tau) \frac{\partial}{\partial \xi} g_2(x, \xi, t - \tau) d\xi \equiv \sum_{j=1}^5 J_{2j}. \end{aligned} \quad (23)$$

The system (19), (21), (22), and (23) is closed by equation (16), which, by virtue of (8), can be written in the form

$$z(t) = 1 + \int_0^t v(\tau) d\tau, \quad (24)$$

and by the equation

$$f(t) = \sum_{i=1}^3 J_i^{(1)}(0, t). \quad (25)$$

We solve the system (19)–(25) by the method of successive approximations. Namely, we set

$$\begin{aligned} f_{n+1}(t) &= \Phi(v_n, q_{1n}, z_n, t); & \dot{F}_{n+1} &= \Psi(v_n, z_n, f_{n+1}, t); \\ v_{n+1}(t) &= V(v_n, z_n, \dot{F}_{n+1}, q_{1n}, q_{2n}, t); & \dot{z}_{n+1} &= v_{n+1}(t); \\ q_{i,n+1}(x, t) &= Q_i(v_{n+1}, z_{n+1}, q_{in}, \dot{F}_{n+1}, x, t). \end{aligned} \quad (26)$$

Here Φ, Ψ, V, Q_1 , and Q_2 are the right-hand sides of equations (25), (21), (19), (22), and (23). As the zero approximations we take arbitrary bounded functions \dot{z}_0, v_0, q_{10} , and q_{20} .

The convergence of the process and the uniqueness of the solution are proved analogously to how this is done in ⁽⁴⁾, on the interval $(0, T)$, depending on M (from (6)) and $|y(0)|^*$.

The equivalence of the constructed solution to the solution of the original problem is established in the following way.

It is proved that the constructed v, f, \dot{F}, q_1 and q_2 are differentiable, with $\dot{v} = w/\sqrt{t}$; $\partial q_i/\partial t = s_i/\sqrt{t}$; $\partial q_i/\partial x = r_i/\sqrt{t}$; $\ddot{F} = F^0/\sqrt{t}$, where w, s_i, r_i and F^0 are bounded for $t \geq 0$. Hence, and from (17), (18), it follows that p_i^* satisfy equations (12)–(16) ($i = 1, 2$) and, in addition,

$$\begin{aligned} \lim_{x \rightarrow 0} p_i^* &= f(t); & \lim_{x \rightarrow 0} q_1 &= \lim_{x \rightarrow 0} \frac{\partial}{\partial x} p_1^* = v(t); & \lim_{x \rightarrow 0} q_2 &= \lambda v(t); \\ \lim_{t \rightarrow 0} p_i^* &= f_i(x) & (i = 1, 2). \end{aligned}$$

Thus, it remains to show that

$$\lim_{x \rightarrow 0} p_2^* = f(t); \quad \lim_{x \rightarrow z(t)} p_2^* = F(t); \quad \lim_{x \rightarrow z(t)} \frac{\partial}{\partial x} p_2^* = \psi(t). \quad (27)$$

Assuming the contrary, we find that there must exist nonzero solutions of the system

$$\mu(t) = 2 \int_0^t \nu(\tau) g_2(z(t), z(\tau), t - \tau) d\tau - 2a^2 \int_0^t \mu(\tau) \frac{\partial}{\partial \xi} g_2(z(t), z(\tau), t - \tau) d\tau; \quad (28)$$

$$\nu(t) = \frac{a}{\sqrt{\pi}} \int_0^t \frac{\mu(\tau)}{\sqrt{t-\tau}} d\tau. \quad (29)$$

Inverting (29) and eliminating $\mu(t)$ from (28), we arrive at the equation

$$\int_0^t \nu(\tau) \bar{K}(t, \tau) d\tau = 0 \quad (30)$$

with kernel $\bar{K}(t, \tau)$, satisfying the conditions

$$\bar{K}(t, t) = \frac{\dot{z}(t)}{2a^2}; \quad \frac{\partial}{\partial t} \bar{K} = O\left(\frac{1}{\sqrt{t-\tau}}\right).$$

Since, by virtue of the second of conditions (7), $\dot{z}(0) \neq 0$, it follows that $\nu(\tau) \equiv 0$, and by virtue of (28) this means that also $\mu(\tau) \equiv 0$.

The contradiction obtained proves the validity of (27) and at the same time completes the proof of the assertion of equivalence.

In conclusion, we note that in an entirely analogous manner one can solve the problem when, on the boundary $\xi = 0$, not $\partial p_2 / \partial \xi$ but p_2 is prescribed. In this case, as above, the assumption $y(0) \neq 0$ is essential.

Krasnodar Branch
of the All-Union Oil and Gas
Scientific-Research Institute

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* The requirement $y(0) \neq 0$ (expressed above by the condition $y(0) = -1$) is essential. Abandoning it would require carrying out an analysis similar to that carried out in solving Stefan's problem ⁽⁵⁾.

Note: Figure translations are in progress. See original paper for figures.

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