ON LEBESGUE-ORLICZ POINTS

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Abstract

Full Text

MATHEMATICS

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ON LEBESGUE-ORLICZ POINTS

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1. In a number of problems in the theory of functions, an important role is played by the relation between the values of a function at a point and the values in a neighborhood of this point. This relation can be expressed in various terms. The role of points of continuity of a function is well known. Another class of points is singled out by the condition

$$\lim_{h\to 0}\frac{1}{2h}\int_{x_0-h}^{x_0+h}|f(x)-f(x_0)|^p\,dx=0\quad(p\geqslant 1). \tag{1}$$

Such points are called Lebesgue points of order p ([1,2]). Points of the first order are simply called Lebesgue points.

Condition (1) can be written in the form

$$\lim_{h\to 0}\left\|M^{(-1)}\left(\frac{1}{2h}\right)[f(x)-f(x_0)]H_h(x)\right\|_M=0, \tag{2}$$

where $H_h(x)$ denotes the characteristic function of the segment $[x_0-h,\ x_0+h]$; $M^{(-1)}(u)$ is the function inverse to the function $M(u)=|u|^p/p$, and the norm is taken in the sense of the space L_p.

The spaces L_p are a special case of a substantially broader class of Orlicz spaces ([1,3]) L^*_M.

Let us recall the basic definitions.

Let $p(u)$ be a right-continuous, nondecreasing function, with $p(+0)=0,\ p(+\infty)=\infty$. Let

$$q(s)=\sup_{p(t)\leqslant s}t.$$

Then the functions

$$M(u)=\int_0^{|u|}p(t)\,dt,\qquad N(v)=\int_0^{|v|}q(t)\,dt$$

are called mutually complementary N'-functions. The Orlicz space L^*_M is defined as the collection of real measurable functions on $[0,1]$ for which the norm is finite:

$$\|f(x)\|_M = \sup_{\int_0^1 N[\psi(x)]\,dx \leqslant 1} \int_0^1 f(x)\psi(x)\,dx.$$

We shall call the point x a **Lebesgue-Orlicz point of order** $M(u)$ for the function $f(x)$, if relation (2) holds.

2. Lebesgue-Orlicz points form an intermediate class between the class of Lebesgue points and the class of points of continuity. This follows from the following theorem.

We shall call a point $x \in [0,1]$ a point of continuity of the function $f(x)$ on a set of full measure if $f(x)$ is continuous at the point x on some set $G \subset [0,1]$ such that $\operatorname{mes} G = 1$.

Theorem 1. *In order that a point x be a point of continuity of a function on a set of full measure, it is necessary and sufficient that x be a Lebesgue–Orlicz point of arbitrary order. Every Lebesgue–Orlicz point of some order is a Lebesgue point.*

The question arises of comparing the classes of Lebesgue–Orlicz points of different orders. In particular, the question is of interest whether every Lebesgue–Orlicz point of order $M(u)$ will be a Lebesgue–Orlicz point of any "lower" order $M_1(u)$, regarding $M_1(u)$ as a lower order if, for large values of the argument, $M_1(u) \leq M(ku)$.

In this direction the following results have been obtained.

Theorem 2. *In order that every Lebesgue point for any bounded function be a Lebesgue–Orlicz point of order $M(u)$, it is necessary and sufficient that $M(u)$ satisfy Young' s Δ_2-condition: $M(2u) \leq kM(u)$ for large values of the argument.*

As is known ([3]), there exists an N'-function $M(u)$ not satisfying the Δ_2-condition, satisfying for large values of the argument the inequality $M(u) \leq u^p$. Then, by Theorem 2, there exists a bounded function having a Lebesgue point of order p which is not a Lebesgue–Orlicz point of order $M(u)$, where $M(u) \leq u^p$ $(p > 1)$.

Theorem 3. *Suppose that, for large values of the argument, $M_2(u) \leq M_1(ku)$. Then, in order that every Lebesgue–Orlicz point of order $M_1(u)$ of any function $f(x)$ be a Lebesgue–Orlicz point of order $M_2(u)$, it is sufficient that two conditions be fulfilled:*

a) *$M_2(u)$ satisfies the Δ'-condition ([4]): for large values of the argument*

$$M_2(uv) \leq M_2(k_2 u) M_2(v);$$

b) *for large values of the argument*

$$M_1(uv) \geq M_1(k_1 u) M_1(v).$$

The last theorem makes it possible to compare Lebesgue points of order p with Lebesgue–Orlicz points of orders $M(u)$, where $M(u)$ are N'-functions growing faster than any power, since for such N'-functions, generally speaking, condition b) is fulfilled. For example, condition b) is satisfied by N'-functions satisfying the Δ^2-condition ([5]). In particular, it follows from Theorem 3 that a Lebesgue –Orlicz point of order $M(u)$ will be a Lebesgue point of any order p, if $M(u) = e^{|u|} - |u| - 1$, $M(u) = e^{u^2} - 1$, etc.

3. From Theorem 3 it follows that the class of Lebesgue–Orlicz points of order $M(u)$ may narrow for rapidly growing N'-functions $M(u)$. The question arises whether it is possible to construct such a function $f(x)$ which has no Lebesgue–Orlicz points of the given order $M(u)$. This problem has a negative solution for the function $M(u) = |u|^p/p$ $(p \geq 1)$, since the set of Lebesgue points of any order has full measure ([1]). Therefore it is natural to consider the case of orders of type $e^{|u|} - |u| - 1$.

Theorem 4. *There exist measurable functions for which no point is a Lebesgue –Orlicz point of order $M(u)$, where $M(u)$ is any N'-function satisfying the Δ^2-condition ([5]): for large values of the argument $M^2(u) \leq M(ku)$.*

The proof of Theorem 4 uses a new concept.

Let $\varphi(h)$ be a function possessing the properties: $0 < \varphi(h) < 1$ for $h > 0$ and $\varphi(h) \to 1$ as $h \to 0$. We shall call a point x_0 a point of density of order $\varphi(h)$ of the set $E \subset [0,1]$ if, for sufficiently small h,

$$\frac{1}{2h} \operatorname{mes}\{[x_0 - h,\ x_0 + h] \cap E\} \geq \varphi(h).$$

It is possible to construct a set $E \subset [0,1]$ such that, for some k, all its points and the points of its complement have density of order not exceeding $\varphi(h) = 1 - kh$. The characteristic function of the set E satisfies the condition of Theorem 4. We indicate here an example of the construction of the set E.

By a **Cantor set** on a segment $[a,b]$ of length $l = b - a$ we shall mean a perfect set obtained by successively deleting central intervals. Namely, from the segment $[a,b]$ we delete an interval of length l/q; from each of the two remaining segments we delete an interval of length l/q^2, and so on. At the n-th step, from each of the 2^n segments we delete an interval of length l/q^{n+1}, and so on. The number q will be called the **denominator** of the Cantor set. The measure of the Cantor set F is related to the denominator q by

$$mF = \frac{(q-3)(b-a)}{q-2} \qquad (q \geq 3).$$

Construct on the interval $[0,1]$ a Cantor set $E^{(1)}$ with denominator $q^{(1)} = 2 + \frac{4}{3}$. Let $\{\alpha_{k^1}^{(1)}\}$ be the intervals complementary to the set $E^{(1)}$. In each segment $\alpha_{k^1}^{(1)}$ $(k^1 = 1, 2, \ldots)$ construct a Cantor set $E_{k^1}^{(2)}$ with denominator

$$q^{(2)} = 2 + \frac{2^2+2}{2^2+1}.$$

Denote

$$E^{(2)} = \sum_{k^1=1}^{\infty} E_{k^1}^{(2)}.$$

Suppose the perfect set $E^{(n)}$ has been constructed. Its complement forms a system of intervals $\{\alpha_{k^n}^{(n)}\}$ $(k^n = 1, 2, \ldots)$. In each of the segments $\alpha_{k^n}^{(n)}$ construct a Cantor set $E_{k^n}^{(n+1)}$ with denominator

$$q^{(n+1)} = 2 + \frac{2^{n+1}+2}{2^{n+1}+1}.$$

Denote by

$$E^{(n+1)} = \sum_{k^n=1}^{\infty} E_{k^n}^{(n+1)}.$$

The set $E^{(n+1)}$ is perfect. We continue this process indefinitely.

Denote

$$E = E^{(1)} + E^{(2)} + \cdots + E^{(n)} + \cdots.$$

The set E has the required property.

4. **Theorem 5.** *In order that the point x_0 be a Lebesgue-Orlicz point of order $M(u)$ of the function $f(x)$, it is necessary and sufficient that there exist a set $E \subset [0,1]$ possessing the following properties:*

a) *the point x_0 is a density point of the set $[0,1] \setminus E$;*

b) *$f(x)$ is continuous at the point x_0 along the set $[0,1] \setminus E$;*

c)

$$\lim_{h\to 0} \left\| M^{(-1)}\left(\frac{1}{2h}[f(x)-f(x_0)]H_h(x)H(x,E)\right)\right\|_M = 0,$$

where $H(x,E)$ is the characteristic function of the set E.

5. Since the Lebesgue-Orlicz points form narrower classes than the Lebesgue points, it is natural to expect that at these points singular integrals will converge under comparatively weak conditions imposed on their kernels. It was precisely in connection with this problem that S. G. Krein proposed generalizing the notion of a multiple Lebesgue point to the case of Orlicz metrics. The study of the convergence of integrals at Lebesgue-Orlicz points can be carried out by the method developed in ([6]).

6. By Lusin' s theorem, for every measurable function $f(x)$ one can indicate sets $E \subset [0,1]$ whose measures are arbitrarily close to one and on which $f(x)$ is continuous. The sets E may differ from one another in their order of density. We shall say that $f(x)$ is continuous with density $\varphi(h)$ if, under the conditions of Lusin' s theorem, one can choose a set E each point of which has density of order $\varphi(h)$. Naturally, the function $f(x)$ should be regarded as the "closer" to being continuous, the greater the density with which it is continuous.

There arises the question of finding the density with which measurable functions are continuous, the question of the connection between the density of continuity of a function and the order of approximation of it in various metrics by polynomials, and so on.

The connection between the notion of a Lebesgue-Orlicz point and the notion of a density point of order $\varphi(h)$, as well as the questions listed in this paragraph, were brought to our attention by M. A. Krasnosel' skii.

The substance of these questions is illustrated by the following assertion, which is a consequence of the proof of Theorem 4.

Theorem 6. *There exists a measurable function $f(x)$ possessing the property that every point of any set on which $f(x)$ is discontinuous has density of order not exceeding $\varphi(h) = 1 - kh$, where k is some number.*

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MATHEMATICS

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ON THE EXISTENCE OF A SOLUTION OF THE PROBLEM OF GENERAL ANAMORPHOSIS

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In 1949 the author of the present note proposed necessary and sufficient conditions for the nomographability of the equation $z = \varphi(x, y)$ by means of a nomogram of aligned points ([5]). The criterion of nomographability was given in the form of relations connecting the partial derivatives of the function $\varphi(x, y)$. An analogous result was obtained by M. A. Kreines and N. D. Eisenstadt ([3,4]) (see ([3]), § 6, p. 350) as one of the consequences of the theory of local nomograms constructed by them.

In the present note the criterion of nomographability is connected more closely with Gronwall' s function, which makes it possible to compute the scales of the nomogram by formulas known already to Gronwall ([2]), a complete list of which is given in ([6]). The use of nomographic invariants ([7,10]) or of invariants of Blaschke nets ([1]) makes it possible to give the computations an easily surveyable form.

Here we shall consider the problem of general anamorphosis for the equation $z = \varphi(x, y)$ with a sufficiently smooth right-hand side, monotone in the two variables x, y, i.e. it will be assumed that in the domain G of definition of the function $\varphi(x, y)$ its first partial derivatives φ_x, φ_y do not vanish. The scales of the nomogram are also assumed to be sufficiently smooth curves. The proposed theory is constructed in a way invariant with respect to regraduation of the scales.

Any sufficiently smooth mapping of the domain G of the plane xOy into a domain $\overline{G}$ of the plane $\overline{xOy}$ of the form

$$\overline{x} = \overline{x}(x), \qquad \overline{y} = \overline{y}(y) \tag{1}$$

such that $\overline{x}'(x) \neq 0$, $\overline{y}'(y) \neq 0$ everywhere in G, will be called a **transformation of regraduation type**.

Let $z = f(x, y)$ be an arbitrary sufficiently smooth and monotone function given in the domain G; $M = -f_y/f_x$.

A polynomial $I(M, 1/M, \ldots, M^{(r)}_{y \ldots y})$ over the field of real numbers in $M, 1/M$ and the partial derivatives of M up to order r inclusive is called an **integral nomographic invariant** if, under any transformation of the domain G of regraduation type which transforms the equation $z = f(x, y)$ into the equation

$z = \overline{f}(\overline{x}, \overline{y})$ and, respectively, $M = -f_y/f_x$ into $\overline{M} = -\overline{f}_{\overline{y}}/\overline{f}_{\overline{x}}$, the identity in x, y

$$I(M, \dots, M^{(r)}_{y \dots y}) = (\overline{x}')^{\alpha}(\overline{y}')^{\beta} I(\overline{M}, \dots, \overline{M}^{(r)}_{\overline{y} \dots \overline{y}}) \quad (2)$$

holds.

The integer α is called the **weight** of I with respect to x; the integer β is called the **weight** of I with respect to y; r is called the **order** of the invariant I. If we are interested only in the weights of the invariant, we write $I = (\alpha, \beta)$. Obviously, I is also an invariant with respect to regraduation of the variable z.

Note: Figure translations are in progress. See original paper for figures.