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Abstract

Full Text

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ON THE COMBINATORIAL TOPOLOGY OF HILBERT SPACE

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This note briefly sets forth a proof of the invariance (under different cell decompositions) of the cohomology groups of infinite-dimensional sets (^{1}). Also defined are the exterior homology groups of an arbitrary closed set F lying in a Hilbert space H.

Theorem. Let some set $P \subset H$ be representable as the closure of the bodies of closed subcomplexes K_1 and K_2 of certain cell decompositions of the Hilbert space H. Then for every r the relation ${}_rH(K_1) \approx {}_rH(K_2)$ holds, where ${}_rH(K)$ is the r-defective cohomology group* of the complex K, defined in note (^{1}), taken with an arbitrary coefficient group.

For the proof we introduce below the notions of a cellular mapping and of the degree of a mapping.

I. Definition of a cellular mapping. Let ${}^rt$ and r_τ be cells of defect r of certain cell decompositions of the Hilbert space H. A continuous mapping $\varphi : {}^rt \to H$ will be called **cellular with respect to the pair of cells** ${}^rt$ and r_τ, if the following two conditions are satisfied:

a) For every number k between zero and one, and every polyhedron M, homothetic to r_τ with homothety coefficient k and lying together with its closure in r_τ, there exists a number $\xi > 0$ such that the set $\varphi({}^rt) \cap (O_\xi M \setminus r_\tau)$ is empty.

b) $\varphi(\dot{r}^t) \subset H \setminus r_\tau$, where by $\dot{r}^t$ is denoted the boundary of the cell rt.

We now fix some orientation sense α of the Hilbert space H (^{1}) and choose two bases of this space, compatible with the orientation sense α: the basis $\{f_i\} = \{f_1, \dots, f_r, f_{r+1}, \dots\}$ and the basis $\{\theta_i\} = \{\theta_1, \theta_2, \dots, \theta_r, \theta_{r+1}, \dots\}$, and we choose these bases so that the vectors $f_{r+1}, f_{r+2}, \dots$ lie in the carrier plane of the cell ${}^rt$, and the vectors $\theta_{r+1}, \theta_{r+2}, \dots$ in the carrier plane of the cell r_τ; from this moment on we shall consider the cells ${}^rt$ and r_τ oriented in the sense α.

We now single out a class of cellular mappings admissible in the sense α, narrow enough so that within it the notion of the degree of a mapping can be meaningfully and correctly defined.

A mapping $\varphi : {}^rt \to H$, cellular with respect to the pair of cells ${}^rt$ and r_τ, will be called **admissible in the sense** α if the following conditions are satisfied.

1°. For any point $x \in rt$, whose image belongs to the cell r_τ, and any positive number ε', there exist a number N' and numbers $\lambda_2 > \lambda_1 > 0$, depending only on x, such that for $n > N'$ the inequalities

$$\lambda_1 \rho({}^{r}t_x^n, {}^{r}t_y^n) - \varepsilon' \leq \rho({}^{r}\tau_{\varphi(x)}^n, {}^{r}\tau_{\varphi(y)}^n) \leq \lambda_2 \rho({}^{r}t_x^n, {}^{r}t_y^n) + \varepsilon',$$

hold.

as soon as $y \in {}^{r}t$, and $\varphi(y) \in {}^{r}\tau$. (By ${}^{r}t_x^n$ is denoted the intersection of the cell ${}^{r}t$ with the Euclidean space $x + H_{\{f_i\}}^{n+r}$, where $H_{\{f_i\}}^{n+r}$ is the linear span of the vectors $f_1, f_2, \dots, f_{n+r}$.)

2°. For each $x \in {}^{r}t$, $\varphi(x) \in {}^{r}\tau$, and each ε'' there exists an N'' such that for $n > N''$ the inequality

$$\rho({}^{r}\tau_{\varphi(x)}^n, \varphi(y)) < \varepsilon''$$

is satisfied as soon as $y \in {}^{r}t_x^n$, and $\varphi(y) \in {}^{r}\tau$; here ${}^{r}\tau_{\varphi(x)}^n$ is understood as the intersection of the cell ${}^{r}\tau$ with the half-space $\varphi(x) + (H_{\{v_i\}}^{r+n-1} \times l)$, where l is the positive ray directed along the vector v_{r+n}.

II. **The degree of a cellular mapping.** Let $x \in {}^{r}t$, $\varphi(x) \in {}^{r}\tau$, and let the distance from the point $\varphi(x)$ to the boundary ${}^{r}\dot{\tau}$ of the cell ${}^{r}\tau$ be equal to 10ρ; then choose k so that any polyhedron M defined in condition a) contains within itself the 4ρ-neighborhood of the point $\varphi(x)$ relative to the cell ${}^{r}\tau$. By the number k, from condition a) one finds a number ξ, which may be assumed smaller than 5ρ. We also choose the numbers ε' and ε'' smaller than $\xi/10(\lambda_1 + \lambda_2)$, if $\lambda_1 + \lambda_2 > 1$, and smaller than $\xi/10$ otherwise. From ε' and ε'', by conditions 1° and 2°, we find N' and N'' and put $N = \max(N', N'')$. We orthogonally project the set $(O_{\xi/2}M) \cap \varphi({}^{r}t_x^N)$ onto ${}^{r}\tau_{\varphi(x)}^N$ and extend this projection in such a way to the intersection of the set $\varphi({}^{r}t_x^N)$ with the set $O_\xi M \setminus O_{\xi/2}M$ that a continuous mapping of the set ${}^{r}t_x^N$ into the space H is obtained; denote the resulting mapping by φ'_N. Since the sets ${}^{r}t_x^N$ and ${}^{r}\tau_{\varphi(x)}^N \cap M$ are convex N-dimensional bodies and, consequently, cells, the degree of the mapping $[\varphi'_N {}^{r}t_x^N : {}^{r}\tau_{\varphi(x)}^N \cap M]$ is defined by virtue of the choice of the mapping φ'_N.

It is proved that, for sufficiently large N, the degree $[\varphi'_N {}^{r}t_x^N : {}^{r}\tau_{\varphi(x)}^N \cap M]$ depends neither on the choice of N nor on the choice of the polyhedron M. It is also proved that, for all points $x \in {}^{r}t$ for which $\varphi(x) \in {}^{r}\tau$, and for sufficiently large N, depending on x, the number $[\varphi'_N {}^{r}t_x^N : {}^{r}\tau_{\varphi(x)}^N \cap M]$ is one and the same. This

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* The terms "cohomology" and "contrahomology" are used in the sense of the previously used terms "lower homology" and "upper homology," respectively.

number we shall call the degree of the mapping φ of the cell ${}^{r}t$ onto the cell ${}^{r}\tau$ and shall denote by $[\varphi {}^{r}t : {}^{r}\tau]$.

III. **Standard extensions.** Choose in the cell τ an arbitrary point p, take its neighborhood O_p relative to the cell τ, having the form $\mu(\tau - p) + p$, where μ is a number between zero and one. Define a deformation $\Pi_\nu : \tau \to \tau$ by the formula:

$$\Pi_\nu = \begin{cases} [\nu\mu^{-1} + (1-\nu)](x-p) + p, & \text{for } x \in \overline{Op}, \\ [\nu\gamma^{-1} + (1-\nu)](x-p) + p, & \text{for } x \in \gamma(\dot{\tau} - p) + p, \end{cases}$$

where $1 \geq \gamma \geq \mu$.

We shall denote the mapping Π_1 by Π_p, or, if this mapping is carried out in several cells, none of which is contained in the geometric boundary of another, by $\Pi_{\{p\}}$, where the index is the set of the corresponding points.

Suppose further that there are given: a number μ, lying between zero and one, a point $p \in \tau$, and a deformation $f_\nu : \tau \to \tau$; the formula

$$f_\nu = \begin{cases} [\nu\mu^{-1} + (1-\nu)](x-p) + p, & \text{for } x \in \overline{Op}, \\ [\nu + (1-\nu)\gamma]\left[f_{\frac{\nu-\mu}{1-\mu}}\left(\frac{x-p}{\gamma} + p\right) - p\right] + p, & \text{for } x \in \gamma(\dot{\tau} - p) + p, \end{cases}$$

where $1 \geq \gamma > \mu$, defines the extension of the deformation f_ν to the whole cell τ; we shall call this extension standard. The mapping f_1 will be called a **standard extension of the mapping** f_1.

IV. **Approximation of the identity mapping.**
Let us now consider two complexes K_1 and K_2, which are subcomplexes of some cell decompositions of the space H. The closures of the point sets –the bodies of these subcomplexes–will likewise be denoted by K_1 and K_2. Suppose also that $K_1 \subset K_2$, and that $i : K_1 \to K_2$ is the inclusion mapping. We shall construct an r-defective cellular approximation i_r of the mapping i.

For each $r = 0, 1, 2, \ldots$ let us number in some way all cells of defect r of the complex K_2, and denote them by ${}^{r}\tau_j$, where r is the defect and j the number. We shall denote the totality of cells of the complex K having the given defect r by ^{rK}, and shall call this totality the r-**defective skeleton** of the complex K.

Construction of the approximation i_0**.** Choose in each cell ${}^{0}\tau_j$ a point ${}^{0}p_j$ so that ${}^{0}p_j \notin i({}^{1}K_1)$ for any j; next choose, for each point ${}^{0}p_j$, a number ${}^{0}\mu_j$ such that the neighborhood ${}^{0}\mu_j({}^{0}\tau_j - {}^{0}p_j) + {}^{0}p$ of the point ${}^{0}p_j$ does not intersect the set $i({}^{1}K_1)$. After this we perform the mapping $\Pi_{\{{}^{0}p_j\}} : K_2 \to K_2$, and denote the mapping $\Pi_{\{{}^{0}p_j\}} i$ by i_0.

Construction of the approximation i_r. Suppose that all the preceding approximations have already been constructed. In each cell ${}^{r}\tau_j$ choose a point ${}^{rp}_{j}$ so that ${}^{rp}_{j} \notin i_{r-1}({}^{r+1}K_1)$ for all j. Also choose, for each point ${}^{rp}_{j}$, a number ${}^{r}\mu_j$ such that the set ${}^{r}\mu_j({}^{r}\tau_j - {}^{rp}_{j}) + {}^{rp}_{j}$ does not intersect the set $i_{r-1}({}^{r+1}K_1)$. After this choice we perform the mapping $\Pi_{\{{}^{rp}_{j}\}} : {}^{rK}_{2} \to {}^{rK}_{2}$, extend it standardly to the skeleton ${}^{r-1}K_2$, then to ${}^{r-2}K_2$, and so on down to the skeleton ${}^{0}K_2$, using the fact that for every cell of each of these skeletons a point p and a number μ have already been chosen. The superposition Πi_{r-1} of the resulting mapping Π and the mapping i_{r-1} we shall call an r-defective cellular approximation of the mapping i and denote it by i_r.

The following assertion is essential: *The mapping i_r is an admissible cellular mapping with respect to any pair of cells ${}^{r}t$ and τ, where ${}^{r}t \in {}^{rK}_{1}$.*

V. **Invariance of infinite-dimensional homologies.**

The constructions carried out above make it possible, in the usual way, to define a homomorphism

$$_*i : {}_rL(K_1) \to {}_rL(K_2)$$

of cochain groups, then to prove commutativity of the coboundary homomorphism with the homomorphism $_*i$, and thereby to construct a homomorphism

$$_*i : {}_rH(K_1) \to {}_rH(K_2)$$

of cohomology groups. For two homomorphisms

$$_*i : {}_rH(K_1) \to {}_rH(K_2)$$

and

$$_*j : {}_rH(K_2) \to {}_rH(K_3)$$

of this kind, generated by the inclusion mappings $i : K_1 \to K_2$ and $j : K_2 \to K_3$, the relation

$$_*(j \circ i) = {}_*j \circ {}_*i$$

holds, expressing the naturality of these homomorphisms; this proves the invariance of the cohomology groups.

VI. **Exterior cohomology groups.**

Let M be a (closed) set lying in the space H. Choose a system of numbers $\varepsilon_i, i = 1, 2, ...,$ decreasing monotonically to zero. Let, further, K_i $(i = 1, 2, ...)$ be a subcomplex of some cell decomposition of the space H, with cell diameter less than $\varepsilon_i/4$, such that

$$O_{\varepsilon_{i+1}}M \subset K_i \subset O_{\varepsilon_i}M,$$

where $O_{\varepsilon_i}M$ is the ε_i-neighborhood of the set M. It is clear that the cohomology groups of defect r of the complexes K_i form an inverse spectrum; the limit of this spectrum will be denoted by

$$_rH\ (M)$$

and will be called the exterior cohomology group of the set M.

The groups ${}_rH\ (M)$ do not depend on the arbitrary elements of the construction.

VII. Analogous results (proved analogously) hold for cohomology groups.

In conclusion, I consider it my duty to express my gratitude to V. G. Boltyanskii for valuable advice and comments.

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References

[1] V. G. Boltyanskii, DAN, **105**, No. 6 (1955).

Note: Figure translations are in progress. See original paper for figures.