# MATHEMATICS

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## Abstract

## Full Text

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# ON “DETERMINING”STATISTICS; A GENERALIZATION OF THE MOMENT PROBLEM

Let $X$ be a one-dimensional random variable with distribution law $F(x) = P(X < x)$, and let $\vec{\xi} = (x_1, x_2, \dots, x_n)$ be the corresponding repeated sample (a random vector with independent components distributed according to the law $F(x)$). Suppose that indirect observations are made on $X$, yielding the values of some continuous statistic $Q(\vec{\xi})$.

It is natural to pose the question: in what cases do observations on the statistic $Q(\vec{\xi})$ help to reconstruct the law $F(x)$; when, for example, do such observations on $Q(\vec{\xi})$ make it possible to construct a reasonable confidence band for $F(x)$? Here we shall deal with the simpler analytic question of when, knowing exactly the distribution law of the statistic

$$F_Q(x) = P(Q < x)$$

one can uniquely reconstruct the law $F(x)$. If, for a given statistic $Q(\vec{\xi})$, this can be done in a class of laws $R$, then we shall say that the statistic $Q(\vec{\xi})$ is “determining” in the given class $R$ of laws $F(x)$.

The problem of investigating determining statistics and the corresponding classes of laws is connected with the classical moment problem and may be regarded as a generalization of that problem. To substantiate these considerations, consider the statistic $Q = \gamma_1 x_1 + \gamma_2 x_2 + \dots + \gamma_n x_n$, where $\gamma_i > 0$ are fixed, and the class $R$ of laws $F(x)$ on the interval $(-\infty, \infty)$ having all moments and uniquely determined by the latter. It is easy to see that the statistic $Q$ will be determining in the class $R$, since the initial moments

$$\alpha_r = \int_{-\infty}^{\infty} x^r \, dF(x)$$

are uniquely expressed in terms of the moments $EQ^r$ of the statistic $Q$. It is also easy to give examples of statistics not determining for the class of all laws $F(x)$ when $n = 2$; the statistic $Q = x_1 - x_2$ will have the same distribution for laws of the form $F_Q(x + \alpha)$ for any $\alpha$.

Here we shall consider homogeneous statistics $Q$ of degree $\mu > 0$ and such that their level surfaces $Q(\vec{\xi}) = c$ are piecewise-smooth star-shaped surfaces situated in a finite region of the sample space (in particular, empty sets). Such statistics will, for brevity, be called definite, and the others indefinite. We shall further say that the parent law $F(x)$ induces the distribution $F_Q(x)$ of the statistic if the sample $\vec{\xi}$ is taken according to this law.

It should be noted that definite statistics will in general not be determining in the class $R$ of all laws $F(x)$; it is easy to give simple examples by

on this matter. However, in broad classes of symmetric laws, definite statistics will be determining; in the present note such phenomena are considered.

Let us introduce here some definitions. We shall say that a set of continuous functions defined on the segment $[a, b]$ forms a class of type $(D)$ if the difference of any two functions of this set either vanishes identically on $[a, b]$, or has there only a finite number of zeros. Examples of classes of type $(D)$ are: polynomials; functions holomorphic at every point of $[a, b]$; quasi-analytic functions of class $(\Delta)$ (1). If the set of functions $y(x)$ forms a class of type $(D)$, and $g(x)$ is some continuous function for which the equation $g(x) = c$, for any $c$, has a finite number of solutions, then the superpositions $y(g(x))$ also form a class of type $(D)$—thus one obtains examples of sufficiently broad classes $(D)$ from nondifferentiable functions. A shift of all functions of a class of type $(D)$ by one and the same continuous function again gives a class of type $(D)$; this makes it possible to immerse any continuous function in a class of type $(D)$.

By a class of type $\{D_{\text{c.s.}}\}$ on the segment $[-\alpha, \alpha]$ $(\alpha < \infty)$ we shall mean a class of functions such that every finite segment $[a, b] \subset [-\alpha, \alpha]$ can be divided into a finite number of segments on which our functions belong to a class of type $(D)$.

Further, a probability density $y(x)$ will be called **fully symmetric** if the corresponding random variable $Z$ can be represented in the form $Z_1 - Z_2$, where $Z_1$ and $Z_2$ are identically distributed random variables having a density belonging to $L_2(-\alpha, \alpha)$, where $[-\alpha, \alpha]$ is the segment on which $y(x)$ is defined.

**Theorem 1.** *In the class of even continuous probability densities of type* $\{D_{\text{c.s.}}\}$*, defined on the segment* $[-\alpha, \alpha]$ *and each having there no more than a countable set of zeros, definite statistics* $Q(\vec{\xi})$ *are determining.*

The following theorem differs somewhat from Theorem 1.

**Theorem 2.** *If an even continuous probability density* $y(x)$ *is immersed in a class of type* $\{D_{\text{c.s.}}\}$ *and has no zeros inside the segment on which it is defined,* $[-\alpha, \alpha]$*, then the distribution of a definite statistic* $Q(\vec{\xi})$*, induced by* $y(x)$*, determines it in this class.*

**Theorem 3.** *Let there be given a fully symmetric density* $y(x)$*, holomorphic at all points of the real axis and inducing the distribution of the definite statistic* $Q(\vec{\xi})$*. Then this distribution determines* $y(x)$ *in the class of all fully symmetric densities of bounded variation in a neighborhood of zero.*

A particular case is the application to the normal law.

**Theorem 3′.** *The normal density* $\frac{1}{\sqrt{2\pi}\sigma}e^{-x^2/2\sigma^2}$ *is determined by the distribution induced by it of any definite statistic in the class of all fully symmetric densities of bounded variation in a neighborhood of* $x = 0$.

Theorems 3 and 3′ rely on a lemma from the theory of positive-definite functions.

**Lemma.** *Let* $g(x)$, $\varphi(x)$ *be even positive-definite functions, with* $\varphi(x)$ *holomorphic in a neighborhood of* $x = 0$, *while this is not assumed for* $g(x)$. *If the difference* $g(x) - \varphi(x)$ *has infinitely many zeros in any neighborhood of* $x = 0$, *then* $g(x) \equiv \varphi(x)$ *on the entire real axis.*

The method of proof of the indicated theorems is the study of certain nonlinear integro-functional equations with shifting arguments; in investigating questions of uniqueness of their solution, they are approximately linearized in the region of small increments of the solutions.

In a similar way one may treat the question of determining the distri-

distribution of the random vector $(X_1, \ldots, X_m)$ by the joint distribution of several statistics.

The behavior of indefinite statistics cannot, generally speaking, be studied by the method by which the preceding theorems were proved, although, as is clear from the example given earlier, many linear statistics will be determining in very broad classes of laws $F(x)$. However, there exists a certain class of indefinite statistics for which theorems analogous to Theorems 1, 2, and 3 can be proved, namely homogeneous statistics of positive dimension whose level surfaces do not touch the coordinate planes. These surfaces must be piecewise smooth, but may consist of several pieces.

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## REFERENCES

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*Note: Figure translations are in progress. See original paper for figures.*

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