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Abstract

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REPRESENTATIONS OF THE ROTATION GROUP OF n-DIMENSIONAL EUCLIDEAN SPACE BY SPHERICAL VECTOR FIELDS

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Let S_n be the sphere in $(n+1)$-dimensional Euclidean space, defined by the equation $\sum_{i=1}^{n+1} x_i^2 = 1$; $R^{(n)}$ is the totality of continuous vector fields tangent to S_n. The purpose of the present note is the decomposition of $R^{(n)}$ into subspaces invariant and irreducible with respect to the group of all rotations S_n, and the indication of the schemes of the corresponding representations.

Let R be an irreducible subspace of $R^{(n)}$. Choose on the sphere a pole–the point $P(0,0,\dots,0,1)$–and denote by R_0 the totality of fields that vanish at the point P. This is a subspace of R, invariant with respect to rotations leaving the point P fixed. Let $\widetilde{R}$ be the orthogonal complement to R_0 in R. (The scalar product of fields u and v from $R^{(n)}$ is defined as $\int_{S_n} (u(M), v(M))\, d\tau$, where $(u(M), v(M))$ is the scalar product of the vectors $u(M)$ and $v(M)$, and $d\tau$ is a measure on S_n invariant with respect to rotations.)

Note that a field from $\widetilde{R}$ is uniquely determined by its value at the point P. Indeed, if $v_1, v_2 \in \widetilde{R}$ and $v_1(P) = v_2(P)$, then $v_1(P) - v_2(P) = 0$, $v_1 - v_2 \in R_0$. At the same time $v_1 - v_2 \in \widetilde{R}$. Consequently, $v_1 - v_2 = 0$; $v_1 = v_2$. The totality of vectors $v(P)$, $v \in \widetilde{R}$, forms a linear subspace in the space R_P of vectors tangent to S_n at the point P. Since it is invariant with respect to all rotations R_P, it either consists of zero alone or coincides with R_P. In the first case, as is easily seen, all of R consists of zero. In the second case $\dim \widetilde{R} = \dim R_P = n$.

Let $\xi_1 = (1,0,\dots,0,0), \dots, \xi_n = (0,0,\dots,1,0)$ be the basis vectors in R_P. The corresponding fields from $\widetilde{R}$ we shall denote by $v_1, v_2, \dots, v_n$ and shall call **zonal fields**. Under all rotations about the point P, zonal fields transform as the coordinates $x_1, x_2, \dots, x_n$.

Now consider the section of S_n by the hyperplane $x_{n+1} = \text{const}$. This is an $(n-1)$-dimensional sphere S_{n-1}; the rotation group S_n leaving the point P fixed coincides on S_{n-1} with the group of all rotations S_{n-1}. The set of zonal fields, if it is considered only on S_{n-1}, generates a space of vector fields on S_{n-1}, invariant with respect to all rotations. These fields will not, generally speaking, be tangent to S_{n-1}. Decompose each field v_k into the sum $v_k' + v_k''$, where v_k' is a field normal to S_{n-1} (and tangent to S_n), and v_k'' is a field tangent to S_{n-1}. At each point of S_{n-1} there exists only one, up to a factor, vector,

normal to S_{n-1} and tangent to S_n. As such a vector we choose $\operatorname{grad}_{S_n} x_{n+1}$. Then

$$v_k' = f_k(x_1, \dots, x_{n+1}) \operatorname{grad}_{S_n} x_{n+1}.$$

Under rotations of S_{n-1} the collection of functions f_k is transformed as a collection of zonal fields, i.e., as a collection of coordinates. But two equivalent collections of spherical functions can differ only by a factor. Therefore $f_k = \lambda x_k$, where λ is constant on S_{n-1}, i.e., $\lambda = \lambda(x_{n+1})$. Finally we obtain

$$v_k' = \lambda(x_{n+1}) x_k \operatorname{grad}_{S_n} x_{n+1}.$$

Returning to the collection of tangent components v_k'', consider the space spanned by it. A zonal collection for this space consists of $n-1$ fields. Consequently, the subspace R_0 in this case is one-dimensional. Let $u \in R_0$. As the pole on S_{n-1} we take the point $P_1(0, 0, \dots, 1, 0)$. As above, decompose u into the sum $u' + u''$, where u' has the form

$$f(x_1, x_2, \dots, x_n) \operatorname{grad}_{S_{n-1}} x_n,$$

and u'' generates on S_{n-2} a one-dimensional space of vector fields invariant under all rotations. But on a sphere of order higher than 1 there are no such spaces. Therefore, for $n > 3$, $u'' = 0$. Further, $f(x_1, \dots, x_n)$ generates on S_{n-2} a one-dimensional space of spherical functions. There is only one such space—the collection of constants. Therefore $f(x_1, \dots, x_n)$ is constant on S_{n-2}, and

$$u' = f(x_n) \operatorname{grad}_{S_{n-1}} x_n = \operatorname{grad}_{S_{n-1}} F(x_n),$$

where $F(x_n)$ is an antiderivative of $f(x_n)$. Thus, $u = u'$ is a gradient field. It follows that the entire space spanned by $\{v_k''\}$ consists of gradient fields. Let

$$v_k'' = \operatorname{grad}_{S_{n-1}} F_k;$$

the functions F_k form a collection of spherical functions on S_{n-1} equivalent to the coordinate collection. Therefore

$$F_k = \lambda x_k, \qquad v_k'' = \lambda \operatorname{grad}_{S_{n-1}} x_k,$$

where λ is constant on S_{n-1}, i.e., $\lambda = \lambda(x_{n+1})$. Finally:

$$v_k = v_k' + v_k'' = \lambda_1(x_{n+1}) x_k \operatorname{grad}_{S_n} x_{n+1} + \lambda_2(x_{n+1}) \operatorname{grad}_{S_{n-1}} x_k.$$

In coordinate notation v_k has the form

$$p \cdot (0, 0, \dots, 0, -x_{n+1}, 0, \dots, 0, x_k) +$$

$$+q\cdot(-x_1x_k, -x_2x_k, \dots, -x_{k-1}x_k, 1-x_k^2-x_{n+1}^2, -x_{k+1}x_k, \dots, x_{nx}k, 0),$$

where

$$p=(1-x_{n+1}^2)\lambda_1; \qquad q=\lambda_2+x_{n+1}\lambda_1.$$

Now let $R\subset R_k^{(n)}$, where $R_k^{(n)}$ is the collection of fields from $R^{(n)}$ whose coordinates are polynomials in $x_1, x_2, \dots, x_{n+1}$ of degree not higher than k. Then p and q in the expression for v_k must be polynomials of degrees not higher than $k-1$ and $k-2$, respectively. The number of linearly independent fields of this form is equal to $2k-1$; and since linearly independent zonal fields correspond to linearly independent irreducible subspaces, $R_k^{(n)}$ decomposes into no more than $2k-1$ irreducible subspaces. In particular, $R_1^{(n)}$ is irreducible. A basis in $R_1^{(n)}$ is formed by fields of the form

$$v_{ij}=(0,\dots,x_j,\dots,-x_i,\dots,0).$$

Hence

$$\dim R_1^{(n)}=C_{n+1}^2.$$

The representation of this dimension is unique (this is easily established using Table 30 in ([1])). We shall denote its highest weight by M_n. The highest weights of the representations realized in spherical functions on S_n, as follows from Cartan' s results ([2]), form a one-dimensional lattice. We shall denote them by $k\Lambda_n$.

We shall show that M_n is not equal to $k\Lambda_n$ for any k. Indeed, in the contrary case the representation of the rotation group of S_n realized in $R_1^{(n)}$ would be equivalent to some representation in spherical functions, and therefore $R_1^{(n)}$ would contain a field invariant with respect to-

subgroup leaving fixed some fixed point. Such a field, as we have seen, must be a gradient field. But the fields belonging to $R_1^{(n)}$ are not gradient fields (if only because the basic fields v_{ij} have closed lines of force), which proves the assertion.

Further, the linear span of the fields $x_iv_j,\ v_j\in R_{k-1}^{(n)}$, belongs to $R_k^{(n)}$. Therefore, if $R_{k-1}^{(n)}$ contains an irreducible component with highest weight N, then $R_k^{(n)}$ contains a component with highest weight $N+\Lambda_n$.

It follows that $R_k^{(n)}$ contains components with highest weights $M_n,\ M_n+\Lambda_n, \dots,\ M_n+(k-1)\Lambda_n$. Moreover, since the gradients of spherical functions of degree k have coordinates of degree $k+1$, $R_k^{(n)}$ contains gradient components with highest weights $\Lambda_n,\ 2\Lambda_n, \dots, (k-1)\Lambda_n$. But the number of components does not exceed $2k-1$; therefore $R_k^{(n)}$ decomposes into the direct sum of the components listed.

Thus, the space $R_\infty^{(n)}$ of fields whose coordinates are polynomials decomposes into irreducible components with highest weights of the form $k\Lambda_n$ and $M_n+k\Lambda_n$.

Now let R be any irreducible subspace of $R^{(n)}$. The corresponding representation is equivalent to at most one of those found. Hence R is orthogonal to all the subspaces found, except possibly one. From this and from the fact that $R^{(n)}_\infty$ is everywhere dense in $R^{(n)}$, it follows that R coincides with one of the subspaces found, or consists of zero alone. This completes the consideration of the case $n > 3$.

n	Schemes of representations by gradient fields and spherical functions on S_n	Schemes of representations by vortex fields on S_n
2	diagram: k, $k \geq 1$	diagram: k, $k \geq 1$
3	diagram: k k, $k \geq 1$	diagrams: $k+2,\ k,$ $k \geq 0$; $k,\ k+2$, $k \geq 0$
4	diagram: k, $k \geq 1$	diagram: $k,\ 2$, $k \geq 0$
5	diagram: k, $k \geq 1$	diagram: $k,\ 1,\ 1$, $k \geq 0$
$2m,\ m \geq 3$	diagram: $k, 0, \dots, 0,$ $k \geq 1$	diagram: $k, 1, 0, \dots, 0,$ $k \geq 0$
$2m+1,\ m \geq 3$	diagram: $k, 0, \dots, 0, 1, 1,$ $k \geq 1$	diagram: $k, 1, 0, \dots, 0, 1, 1,$ $k \geq 0$

The case $n = 3$ is investigated similarly and differs only in that on a sphere of order 1—a circle—there exists a tangent field invariant under rotations. Therefore, in the decomposition $u = u' + u''$ one has to take into account the tangent component u''. This leads to the fact that $R^{(3)}_k$ decomposes into $3k-1$ components (instead of $2k-1$ for $n > 3$).

In particular, $R^{(3)}_k$ decomposes into two three-dimensional components with highest weights M'_3 and M''_3, while all of $R^{(3)}$ decomposes into components with highest weights $k\Lambda_3,\ k\Lambda_3 + M'_3,\ k\Lambda_3 + M''_3$.

The case $n = 2$ could have been investigated similarly, but it is simpler to use the fact that every field tangent to S_2 is representable in the form $\operatorname{grad}_{S_2} f + J \operatorname{grad}_{S_2} g$, where J is the operator of vector multiplication by the outward normal, and f and g are functions on the sphere. It follows at once that all irreducible subspaces of $R^{(2)}$ have the form $\{\operatorname{grad}_{S_2} f_i\}$ or $\{J \operatorname{grad}_{S_2} f_i\}$, where f_i are elements of an irreducible subspace of spherical functions on S_2.

In conclusion we give the schemes of the representations realized in spherical vector fields. (Following E. B. Dynkin[1], we specify the scheme of a representation by writing above a simple root α the number $\Lambda_\alpha = \dfrac{2(\Lambda, \alpha)}{(\alpha, \alpha)}$, where Λ is the highest weight of the representation. Zero values of Λ_α are omitted.)

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REFERENCES

[1] E. B. Dynkin, *Tr. Mosk. matem. obshch.* **1** (1952).
[2] E. Cartan, *Rend. d. Circolo mat. di Palermo*, **53** (1929).

Note: Figure translations are in progress. See original paper for figures.