Physics

1957

SovietRxiv

Abstract

Full Text

Physics

S. I. Braginskii

On the Types of Plasma Oscillations in a Magnetic Field

(Presented by Academician M. A. Leontovich, 18 I 1957)

1. Basic equations. Small oscillations of a homogeneous plasma in a magnetic field have repeatedly been considered both in the hydrodynamic approximation ([1–12]) and with the aid of the kinetic equation (see, for example, ([13])). Nevertheless, it seems useful to consider, in a unified way, oscillations over the entire frequency range for an arbitrary direction of propagation and wavelength, using a model in the form of two charged ideal gases—electron and ion. This makes it possible to systematize the various types of oscillations and also to obtain some new results. We shall start from the system of linearized hydrodynamic equations for electrons and ions

$$\dot{\rho}_\alpha + \rho_\alpha^0 \operatorname{div} \mathbf{v}_\alpha = 0,$$
$$\rho_\alpha^0 \dot{\mathbf{v}}_\alpha = e_\alpha n_\alpha^0 \left(\mathbf{E} + \frac{1}{c} [\mathbf{v}_\alpha \mathbf{H}_0] \right) - \nabla p_\alpha. \tag{1}$$

The index α denotes the species of particles: $\alpha = e$—electrons, $\alpha = i$—ions. The unperturbed densities, particle concentrations, and external magnetic field are denoted by the subscript zero. The oscillating quantities—the densities ρ_α, velocities $\mathbf{v}_\alpha$, pressures p_α, and fields $\mathbf{E}, \mathbf{H}$—are assumed to be small. The unperturbed plasma is neutral. We shall assume—and this is the principal crude assumption of the theory under consideration—that the pressure perturbation is a scalar, related to the density perturbation by the usual ideal-gas relation

$$p_\alpha = c_\alpha^2 \rho_\alpha, \tag{2}$$

where $c_\alpha^2 = \partial p_\alpha / \partial \rho_\alpha = \gamma_\alpha T_\alpha^0 / m_\alpha$; T_α^0 is the temperature; m_α is the mass of particles of species α; γ_α is a coefficient of order unity, having the meaning of an (effective) adiabatic exponent. Equations (1) are to be solved together with Maxwell' s equations.

Consider a plane monochromatic wave. In it all quantities vary proportionally to $e^{i(\mathbf{kr} - \omega t)}$, so that in the equations one may replace $\partial / \partial t$ by $-i\omega$ and ∇ by $i\mathbf{k}$. Eliminating, with the aid of the time-independent Maxwell equations, two quantities (for example $(\mathbf{kE})$ and $(\mathbf{kH})$), we obtain for the remaining 12 quantities $(\rho_i, \rho_e, \mathbf{v}_i, \mathbf{v}_e, [\mathbf{kE}], [\mathbf{kH}])$ a system of homogeneous equations of the form $i\omega f = ...$, where the right-hand sides do not contain the frequency. Setting the determinant of the system equal to zero, we obtain the dispersion equation. It is of degree 12 in ω, but since we do not take into account any dissipative processes, the equations used are reversible; therefore the dispersion equation contains not ω directly, but ω^2. Thus, in the general case we obtain 6 branches of the function $\omega^2(\mathbf{k})$. In practice it is more convenient to arrange the calculations somewhat differently. Instead of the velocities of the ions and electrons, we introduce the mass velocity $\mathbf{v}$ and the relative velocity $\mathbf{u} = \mathbf{v}_i - \mathbf{v}_e$. All quantities are easily expressed in terms of $\mathbf{u}$ and $(\mathbf{kv})$,

for which the equations are obtained

$$(\omega^2 - \omega_0^2 - c_e^2 k^2)\mathbf{u}_\parallel + \frac{\omega^2(\omega^2 - \omega_0^2 - c^2k^2)}{\omega^2 - c^2k^2}\mathbf{u}_\perp - i\omega\Omega_e[\mathbf{hu}] + i\omega\Omega_e[\mathbf{hv}_\parallel] + \mu\Omega_e^2\{\mathbf{h}[\mathbf{hu}]_\perp\} + c_e^2k^2\mathbf{v}_\parallel = 0; \tag{3}$$

$$(\omega^2 - c_0^2k^2)\mathbf{v}_\parallel + \mu i\omega\Omega_e[\mathbf{hu}]_\parallel + \mu c_e^2k^2\mathbf{u}_\parallel = 0. \tag{4}$$

Here $\mathbf{h} = \mathbf{H}_0/H_0$. The sign $\perp$ denotes components of vectors perpendicular to $\mathbf{k}$, and the sign $\parallel$ denotes parallel components. We have introduced the notation

$$\omega_0^2 = 4\pi e^2 n_e/m_e; \qquad \Omega_e = eH_0/m_{ec}; \qquad c_0^2 = c_e^2\rho_e^0 + c_i^2\rho_i^0; \qquad \mu = \rho_e^0/\rho_i^0. \tag{5}$$

In the coefficients of the equations we have neglected terms of order μ as compared with unity. We shall choose the coordinate axes so that the vector $\mathbf{k}$ is directed along the x-axis and so that $h_y = 0$.

2. High frequencies. To simplify the subsequent calculations we shall assume that the electron Larmor frequency is small compared with the Langmuir frequency:

$$\Omega_e \ll \omega_0. \tag{6}$$

This assumption is valid if the magnetic energy does not greatly exceed the particle energy: $\Omega_e^2/\omega_0^2 = (H_0^2/4\pi\rho_e c_e^2)(c_e^2/c^2) \ll 1$.

Condition (6) leads to the fact that, of the six roots of the dispersion equation, three turn out to be of a larger order of magnitude than the other three. In the high-frequency oscillations the motion of the ions may be neglected, and we obtain them from equation (3), putting in it $\mu = 0$, $\mathbf{v} = 0$. The dispersion equation takes the form

$$(\omega^2-\omega_e^2)(\omega^2-\omega_c^2)^2-\Omega_e^2\left[h_x^2\omega^{-2}(\omega^2-\omega_e^2)(\omega^2-c^2k^2)^2+h_z^2(\omega^2-\omega_c^2)(\omega^2-c^2k^2)\right]=0, \tag{7}$$

where

$$\omega_e^2=\omega_0^2+c_e^2k^2; \tag{8}$$

$$\omega_c^2=\omega_0^2+c^2k^2. \tag{9}$$

Equation (7) is of fourth degree in ω^2, but only three roots are determined by it correctly. The fourth root is small ($\sim \Omega_e^2$); it is not in all cases correctly determined by equation (7) and will be considered below. In order to obtain simple explicit expressions for the three "large" roots, we consider the limiting cases of small and large wavelengths, the role of the characteristic length being played by

$$a=(c^2/\omega_0\Omega_e)^{1/2}. \tag{10}$$

A. Short waves. For $a^2k^2 \gg 1$ equation (7) has the roots

$$\omega^2=\omega_e^2+\Omega_e^2h_z^2(1-\omega_e^2/c^2k^2); \tag{11}$$

$$\omega^2=\omega_c^2\pm\omega_0\Omega_e h_x(1+c^2k^2/\omega_0^2)^{-1/2}. \tag{12}$$

The root (11) corresponds to longitudinal electron plasma waves. Their energy is concentrated mainly in the electric field, in the kinetic energy of the electrons, and in their pressure. Electromagnetic waves in the plasma (for example, light) have the dispersion (12). One of the roots (12) corresponds to oscillations having right circular polarization,

the other—left-handed. The energy of the oscillations is contained in the electromagnetic field and in the kinetic energy of the electrons.

B. Long waves. For $a^2k^2 \ll 1$ the roots of (7) are

$$\omega^2=\omega_e^2+h_z^2c^2k^2; \tag{13}$$

$$\omega^2=\omega_c^2\pm\omega_0\Omega_e. \tag{14}$$

In oscillations occurring with frequency (13), the electrons move chiefly along the magnetic lines of force, while in oscillations (14) they move in the plane perpendicular to the magnetic lines of force; in this plane one of the oscillations (14) has right circular polarization, and the other left. In all three cases the energy of the oscillations is concentrated mainly in the electric field and in the kinetic energy of the electrons.

3. Low frequencies. We find the low-frequency branches by making the assumption (confirmed by the result) that $\omega^2 \ll \omega_0^2$, $\omega^2 \ll c^2k^2$. Projecting (3) onto the x-axis, it is easy to see that u_x is small. The low-frequency oscillations are therefore quasineutral; to sufficient accuracy they may be considered by neglecting the volume charge and the displacement current. Let

$$\varkappa = \omega_c^2/c^2k^2 = 1 + \omega_0^2/c^2k^2; \tag{15}$$

$$\Omega_0^2 = \mu\Omega_e^2/\varkappa. \tag{16}$$

The dispersion equation has the form

$$(\omega^2 - \Omega_0^2 h_x^2)\,[\omega^4 - \omega^2(\Omega_0^2 + c_0^2k^2) + \Omega_0^2 h_x^2 c_0^2 k^2] - (\mu\varkappa)^{-1}\Omega_0^2 h_x^2 \omega^2(\omega^2 - c_0^2k^2) = 0. \tag{17}$$

Exact simple expressions for the roots of (17) are obtained if $h_x = 1$ or $h_x = 0$.

For propagation of the wave along the magnetic field ($h_x = 1$) we have

$$\omega^2 = c_0^2k^2; \qquad \omega^2 = \frac{\Omega_e^2}{\varkappa^2}\left(\mu\varkappa + \frac{1}{2} \pm \sqrt{\mu\varkappa + \frac{1}{4}}\right). \tag{18}$$

For propagation across the magnetic field ($h_x = 0$) we have

$$\omega^2 = c_0^2k^2 + \Omega_0^2; \qquad \omega^2 = 0; \qquad \omega^2 = 0. \tag{19}$$

To find the frequencies for an arbitrary direction of propagation, consider two limiting cases.

A. Long waves. Let $\mu\varkappa \gg 1$. In this case $\omega_0^2/c^2k^2 \gg 1$, so that this condition is equivalent to $b^2k^2 \ll 1$, where

$$b^2 = \rho_i c^2/4\pi e^2 n_e^2. \tag{20}$$

The roots of the dispersion equation (17) are approximately equal to

$$\omega^2 = \Omega_0^2 h_x^2 \simeq c_A^2(hk)^2, \qquad \text{where } c_A^2 = H_0^2/4\pi\rho_0; \tag{21}$$

$$\omega^2 = \frac{1}{2}\left[c_A^2 + c_0^2 \pm \sqrt{c_A^4 + c_0^4 + 2c_A^2 c_0^2(1-2h_x^2)}\right]k^2. \tag{22}$$

The first of these roots corresponds to Alfvén waves, the other two to magnetosonic waves, which were investigated in ([5–8]).

B. Short waves. Let $\mu\varkappa \ll 1$; in this case it may be that $\omega_0^2/c^2k^2 \gtrless 1$. In this case the roots of the dispersion equation are approximately equal to

$$\omega^2 = \mu\Omega_0^2 h_x^2 = \Omega_i^2 h_x^2, \qquad \text{where } \Omega_i = e_{iH}0/m_i c; \tag{23}$$

$$\omega^2 = \frac{1}{2}\left[\Omega_0^2 h_x^2/\mu\varkappa + \Omega_0^2 + c_0^2k^2 \pm \sqrt{\left(\Omega_0^2 h_x^2/\mu\varkappa + \Omega_0^2 + c_0^2k^2\right)^2 - 4c_0^2k^2\Omega_0^2h_x^2/\mu\varkappa}\right]. \tag{24}$$

The frequency (23) corresponds to waves that may be called ion gyromagnetic waves. The energy of these waves is concentrated mainly in the kinetic energy of the ions, which move along circles in a plane perpendicular to the wave vector (and not to the magnetic field), so that the condition of quasineutrality of the plasma is fulfilled. In a weak magnetic field ($\Omega_e \ll \varkappa^2 c_0^2 k^2$), one of the roots of (24) corresponds to waves close to sound waves, $\omega^2 \simeq c_0^2k^2$ (formula (24) gives no corrections to $c_0^2k^2$). The other root, $\omega^2 = \Omega_e^2 h_x^2/\varkappa^2$, corresponds to oscillations in which the vectors $\mathbf{H}$ and $\mathbf{u}$ are parallel and have circular polarization. The energy of these oscillations, for $\omega_0^2/c^2k^2 \gg 1$, is concentrated mainly in the magnetic field, while for $\omega_0^2/c^2k^2 \lesssim 1$ it is in the magnetic field and in the kinetic energy of the electrons; thus it is natural to call these oscillations magneto-electron oscillations. In a strong field ($\Omega_e^2 \gg \varkappa^2 c_0^2 k^2$), for $h_x^2 \gg \mu\varkappa$, likewise one of the roots $\omega^2 \simeq c_0^2k^2$ corresponds to sound waves, and the other, $\omega^2 \simeq \Omega_e^2 h_x^2/\varkappa^2$, to magneto-electron oscillations. If, however, the direction of propagation is very close to the direction of the magnetic field ($h_x^2 \ll \mu\varkappa \ll 1$), then the first root tends to zero, and the second to $\Omega_0^2 + c_0^2k^2$. Both oscillations then become considerably more complicated, with both ions and electrons taking an essential part in them.

Short magnetohydrodynamic waves differ substantially from long ones. This shows that the usual equations of magnetic hydrodynamics, which lead (for infinite conductivity) to the picture of "frozen-in" magnetic lines of force and to expressions (21), (22), are invalid for wavelengths smaller than b.

It is of interest to compare b with the mean free path of the particles. Putting $l = 1/n_0\sigma$, we obtain $l/b = (4\pi e^2/\rho_i c^2\sigma^2)^{1/2}$. If one takes $\sigma \sim 10^{-14}$ cm—the cross section of Coulomb collisions at a temperature of several electron volts—then for a hydrogen plasma this gives $l/b \sim (10^{13}/n_0)^{1/2}$. Thus both cases may occur: in a dense plasma $l < b$, and in a rarefied one $l > b$.

Let us compare the possible types of oscillations in the presence and in the absence of an external magnetic field. For $H_0 = 0$, longitudinal electron waves ($\omega = \omega_e$), transverse electromagnetic waves with two independent directions of polarization (the frequency $\omega = \omega_c$ is doubly degenerate), and sound waves ($\omega = c_0 k$) are possible in a plasma. In addition, stationary slip motions are possible in directions perpendicular to the wave vector ($\omega = 0$). The frequency $\omega = 0$ is fourfold degenerate, since there are two kinds of particles and two independent directions of slip. In a magnetic field such slip motions are possible only for $\mathbf{k} \perp \mathbf{H}$; for other directions of $\mathbf{k}$ they are "twisted" by the magnetic field and give two additional types of oscillations (two branches of ω^2). The first four branches are also distorted; in particular, the degeneracy of the frequency ω_c is lifted.

The author thanks A. Kazantsev for a useful discussion.

Received
20 XII 1956

REFERENCES

1. V. L. Ginzburg, *Theory of the Propagation of Radio Waves in the Ionosphere*, 1949.
2. H. Alfvén, *Cosmical Electrodynamics*, IL, 1952.
3. E. Aström, *Nature*, **165**, 1019 (1950); *Ark. f. Fys.*, **2**, No. 42, 443 (1950).
4. H. Herlofson, *Nature*, **165**, 1020 (1950).
5. H. K. van de Hulst, in the collection *Problems of Cosmic Aerodynamics*, IL, 1953, p. 58.
6. V. L. Ginzburg, ZhETF, **21**, 788 (1951).
7. A. Schluter, in the collection *Plasma Dynamics*, IL, 1956, p. 80.
8. A. Baños, *Phys. Rev.*, **97**, 1435 (1955); *Proc. Roy. Soc.*, **A 233**, 350 (1955).
9. V. C. A. Ferraro, *Proc. Roy. Soc.*, **A 233**, 310 (1955).
10. L. Spitzer, *Physics of Fully Ionized Gases*, N. Y., 1956.
11. B. N. Gershman, ZhETF, **31**, 707 (1956).
12. J. H. Piddington, *M. N. Roy. Astr. Soc.*, **114**, 638 (1954).

13. B. N. Gershman, ZhETF, **24**, 453 (1953).

Note: Figure translations are in progress. See original paper for figures.