MATHEMATICS

1957

SovietRxiv

Abstract

Full Text

MATHEMATICS

A. G. KOSTYUCHENKO

ON THE BEHAVIOR OF EIGENFUNCTIONS OF SELF-ADJOINT OPERATORS

(Presented by Academician S. L. Sobolev on 10 XII 1956)

1. In recent years many authors have investigated the question of the behavior at infinity of eigenfunctions of differential operators defined in the whole space. Chiefly the Sturm—Liouville equation on the whole line or half-line was considered. The most complete results were obtained by E. E. Schnol ([1,2]), who showed that almost all, in the corresponding sense, eigenfunctions of the equation

$$-y'' + q(x)y = \lambda y, \qquad -\infty < x < \infty, \qquad q(x) > c > -\infty, \tag{A}$$

grow no faster than $a|x|^{1/2+\varepsilon}$. For the Schrödinger equation $-\Delta u + qu = \lambda u$, $q > -\infty$, he showed that the eigenfunctions $u(x, \lambda)$ belonging to the discrete spectrum decrease exponentially. How eigenfunctions behave in the continuous spectrum remained unknown. However, a conjecture was put forward, based on quantum-mechanical considerations, namely that in the case (A) almost all eigenfunctions should simply be bounded.

In the present note one general theorem is proved, from which the validity of the indicated conjecture follows at once.

2. Denote by L_2 the space of functions $f(x)$, $x = (x_1, x_2, \dots, x_n)$, square-summable over the whole space R_n. Let a self-adjoint operator A be given in L_2. Denote by E_λ its resolution of the identity.

It is known that the whole space L_2 can be decomposed into a direct sum of subspaces $L_2^{(\alpha)}$, and in each of them the operator A will have simple spectrum. This means that in each subspace $L_2^{(\alpha)}$ there is a generating vector $g^{(\alpha)}$, i.e. one such that the closed linear span of the vectors $E_\lambda g^{(\alpha)}$ coincides with $L_2^{(\alpha)}$. Denote by $\sigma_\alpha(\lambda) = (E_\lambda g^{(\alpha)}, g^{(\alpha)})$ the spectral measures of the operator A. It is known that in order to obtain a complete system of eigenfunctions of the operator A one must differentiate the function $E_\lambda g^{(\alpha)}(x)$ with respect to the measure $\sigma_\alpha(\lambda)$ ([3,4]). Such a derivative always exists, but is, generally speaking, a generalized function ([5]).

If the resolvent R_{λ_0} of the operator A, at least for one λ_0, is an integral operator with a Carleman kernel, then the derivative $dE_\lambda g^{(\alpha)}(x)/d\sigma_\alpha(\lambda)$ is an ordinary function ([3,4]).

Theorem 1. Let, for at least one λ_0 (possibly complex), the resolvent of the operator A be an integral operator

$$R_{\lambda_0} f = \int_{-\infty}^{\infty} K(x,y) f(y)\, dy$$

with kernel $K(x,y)$, which for almost all x satisfies the condition

$$\int_{-\infty}^{\infty} |K^2(x,y)|\, dy \leq C, \tag{a}$$

where the constant C does not depend on x. Then almost all, with respect to the measure $\sigma_\alpha(\lambda)$, eigenfunctions $dE_\lambda g^{(\alpha)}/d\sigma_\alpha(\lambda)$ are bounded in x.

Proof. Denote $E_{\lambda_{i+1}} g^{(\alpha)}(x) - E_{\lambda_i} g^{(\alpha)}(x)$ by $E_{\Delta_i} g^{(\alpha)}(x)$. We shall show that, for any partition of the λ-axis, the inequality

$$\sum |E_{\Delta_i} g^{(\alpha)}(x)| < M \tag{1}$$

holds. Here the constant M does not depend on x or on the partition.

Let ε_i denote $+1$ or -1, and let $G^{(\alpha)}_{\Delta_i}(x) = (A - \lambda_0 E)^{-1} E_{\Delta_i} g^{(\alpha)}(x)$. Then, for any choices of the signs ε_i, we have:

$$\sum_i \varepsilon_i E_{\Delta_i} g^{(\alpha)}(x) = \sum \varepsilon_i \int_{-\infty}^{\infty} K(x,y) G^{(\alpha)}_{\Delta_i}(y)\, dy = \int_{-\infty}^{\infty} K(x,y) \left[\sum \varepsilon_i G^{(\alpha)}_{\Delta_i}(y)\right] dy.$$

By the Cauchy–Bunyakovsky inequality one may write:

$$\sum_i \varepsilon_i E_{\Delta_i} g^{(\alpha)}(x) \leq \left(\int_{-\infty}^{\infty} |K^2(x,y)|\, dy\right)^{1/2} \left(\int_{-\infty}^{\infty} \left[\sum_i \varepsilon_i G^{(\alpha)}_{\Delta_i}(y)\right]^2 dy\right)^{1/2}; \tag{2}$$

since for $j \neq i$ the vectors $E_{\Delta_j} g^{(\alpha)}$ and $E_{\Delta_i} g^{(\alpha)}$ are orthogonal, inequality (2) can be written in the form

$$\sum_i \varepsilon_i E_{\Delta_i} g^{(\alpha)}(x) \leq C \left(\int_\lambda (\lambda - \lambda_0)^2\, d\sigma_\alpha(\lambda)\right)^{1/2} \leq C C_1^*.$$

Since the signs ε_i are chosen arbitrarily, inequality (1) is proved.

Inequality (1) shows that $E_\lambda g^{(\alpha)}$ may be regarded as a functional in the space L_1 of summable functions, and the family of functionals $E_\lambda g^{(\alpha)}(x)$ will have strongly bounded variation. Since the function $(E_\lambda g^{(\alpha)}, \varphi)$ is absolutely continuous with respect to the measure $\sigma_\alpha(\lambda)$ for any finite function φ, it follows from a lemma of I. M. Gel' fand ([6]) that $E_\lambda g^{(\alpha)}$ can be differentiated with respect to the measure $\sigma_\alpha(\lambda)$ as a functional in L_1, and the derivative $dE_\lambda g^{(\alpha)}/d\sigma_\alpha(\lambda)$ will also be a linear functional in L_1. But every linear functional in L_1 is given by a bounded measurable function. Therefore the eigenfunctions obtained will be bounded in x. At the same time we have proved that the derivative is an ordinary function.

Remark 1. Instead of requiring the existence of the resolvent with condition (a), one could have required that, for some m, the operator $(A-\lambda_0 E)^{-m}$ be an integral operator with kernel satisfying the same condition.

Remark 2. If it is known that the kernel $K(x, y)$ has the property that

$$\int_{-\infty}^{\infty} |K^2(x,y)|\, dy < M(r), \quad r = |x|,$$

where $M(r)$ is a monotone function of r, then it is shown analogously that the eigenfunctions grow no faster than $aM(r)$ (a is a certain constant).

Remark 3. The eigenfunctions $dE_\lambda g^{(\alpha)}/d\sigma_\alpha(\lambda)$, generally speaking, will not be uniformly bounded in λ. However, it is not difficult to show that, for any number N, there exists a set S_N of measure less than $1/N$ such that, if the set S_N is removed from the spectrum of the operator A, then on the remaining part the eigenfunctions will be bounded uniformly also in λ.

* The integral on the right-hand side of inequality (2) exists, since $g^{(\alpha)}$ belongs to the domain of the operator

3. If at the point λ_0 the resolvent R_{λ_0} is an integral operator with a Carleman kernel, then it is easy to show[3] that $E_\lambda f$ is also an integral operator with kernel $\vartheta(x, y, \lambda)$. The kernel $\vartheta(x, y, \lambda)$, called the spectral function, is defined by the formula

$$\vartheta(x, y, \lambda) = (A - \bar{\lambda}_0 E)(E_\lambda - E_0)K(x, y).$$

If the resolvent kernel $K(x, y)$ satisfies condition (a), then, just as above, one can establish that for every interval of the spectrum $[m, M]$ the inequality

$$\sum |\vartheta(x, y, \lambda_{i+1}) - \vartheta(x, y, \lambda_i)| < C. \tag{3}$$

holds.

Here the constant C depends only on m and M.

Inequality (3) shows that $\vartheta(x, y, \lambda)$, in the space L_1 of summable functions $\varphi(x, y)$ of the variables x, y, generates a family of linear functionals having strongly bounded variation. Since the spectral function $\vartheta(x, y, \lambda)$ is absolutely continuous with respect to the corresponding spectral measure $\sigma(\lambda)$, $\vartheta(x, y, \lambda)$ can be differentiated as a functional in L_1 with respect to the measure $\sigma(\lambda)$, and the derivative $d\vartheta(x, y, \lambda)/d\sigma(\lambda) = \psi(x, y, \lambda)$ will likewise be a functional in L_2. Thus the following theorem has been proved.

Theorem 2. *If the operator A has a resolvent with property (a), then the spectral kernel $\psi(x, y, \lambda)$, for almost all λ with respect to the measure $\sigma(\lambda)$, is bounded in the aggregate of the variables x, y.*

Remarks 1–3 also apply to Theorem 2.

4. As an example, consider the Schrödinger equation in the whole space R_3:

$$-\Delta u + q(x)u = \lambda u, \qquad x = (x_1, x_2, x_3), \qquad q(x) > c_0 > -\infty.$$

It is known that the resolvent of this operator is an integral operator with a Carleman kernel[3]. It is easy to show[2] that for $-\lambda_0 + c_0 = a > 0$ the kernel $K(x, y, \lambda_0)$ satisfies the inequality

$$K(x, y, \lambda_0) < \frac{e^{-\sqrt{a}\, r}}{r}, \qquad r = |x - y|.$$

Hence condition (a) is fulfilled.

Thus the following theorem has been proved.

Theorem 3. *If $q(x) > c_0$, then, with respect to the measure $\sigma_\alpha(\lambda)$, almost all eigenfunctions $u^{(\alpha)}(x, y)$ of the Schrödinger equation $-\Delta u + qu = \lambda u$ are bounded. The spectral kernel $\psi(x, y, \lambda)$ will also be bounded in the aggregate of the variables x, y for almost all λ with respect to the measure $\sigma(\lambda)$.*

In the case $n = 1$, this theorem is a strengthening of the theorem of E. E. Shnol' mentioned above. If $n > 3$, then the theorem is also valid. In this case one must use Remark 1.

Let us note that E. E. Shnol' constructed an example of a self-adjoint Sturm-Liouville differential operator with $q(x)$ unbounded below, whose eigenfunctions in x are unbounded[2].

I take this opportunity to express my deep gratitude to I. M. Gel'fand for posing the problem and for his constant interest in the work.

Moscow State University
named after M. V. Lomonosov

Received
7 XII 1956

REFERENCES

1. E. E. Shnol' , *Uspekhi Mat. Nauk*, **9**, no. 4 (1954).

2. E. E. Shnol' , Dissertation, Moscow, 1955.

3. A. Ya. Povzner, *Mat. Sbornik*, **32** (74), no. 1, 109 (1953).

4. F. Mautner, *Proc. Nat. Acad. Sci.*, **39**, 49 (1953).

5. I. M. Gel' fand, A. G. Kostyuchenko, *Dokl. Akad. Nauk SSSR*, **103**, no. 3 (1955).

6. I. M. Gel' fand, *Mat. Sbornik*, **4**, 46 (1938).

Note: Figure translations are in progress. See original paper for figures.