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Abstract

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MATHEMATICS

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ON SOME OPERATORS IN GENERALIZED ORLICZ SPACES

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The application of topological methods to the study, in functional spaces, of nonlinear Hammerstein integral equations

$$u(x) = \Gamma u = \int_B K(x,y) g(u(y),y)\,dy$$

usually uses the complete continuity of the Hammerstein operator Γ. Since this operator is the product of the linear integral operator

$$Au = \int_B K(x,y) u(y)\,dy$$

and the operator $hu = g(u(x), x)$, which is called [1] the Nemytskii operator, its complete continuity follows from the continuity and boundedness of the operator h and the complete continuity of the operator A. In the case, however, when A is not completely continuous, Γ also usually does not possess this property; but if, in this case, Γ is weakly continuous,* then another approach is possible, using the weak topology. The weak continuity of the operator $\Gamma = Ah$ is ensured by the boundedness of A and the weak continuity of h. In view of this it is important to clarify conditions for the weak continuity of the operator h.

In the present paper, under certain restrictions, necessary and sufficient conditions are found for the weak continuity of the Nemytskii operator in Orlicz spaces. As a preliminary matter, the multiplication operator H, $Hu = b(x)u(x)$, is studied in these spaces.

By Orlicz spaces we mean here generalized Orlicz spaces in the sense of Zaanen [2,3]. These spaces include all spaces L^p $(1 \le p \le \infty)$, whereas the usual Orlicz spaces [4–6] exclude the cases $p = 1$ and $p = \infty$.

1. We recall some definitions concerning generalized Orlicz spaces. Let $\varphi(u)$ $(0 \leq u < \infty)$ be a real function taking only finite values and satisfying the following conditions: $\varphi(0) = 0$; $\varphi(u)$ is nondecreasing, continuous from the left, and is not identically equal to zero (in the usual interpretation one additionally requires that $\varphi(u) > 0$ for $u > 0$ and $\lim_{u\to\infty} \varphi(u) = \infty$). Further, let

$$\psi(v) = \sup\{u \geq 0 : \ \varphi(u) < v\} \quad (0 < v < \infty)$$

and $\psi(0) = 0$. The function $\psi(v)$ satisfies the same conditions as $\varphi(u)$, but may take infinite values. Namely, if $\lim_{u\to\infty} \varphi(u) = l < \infty$, then $\psi(v) = \infty$ for

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* An operator is called **weakly continuous** from E into E_1 (E and E_1 are Banach spaces) if it maps E into E_1 continuously with respect to the weak topologies of these spaces.

$v > l$ or when $v \geqslant l$. The functions

$$\Phi(u) = \int_0^u \varphi(t)\, dt \quad (u \geqslant 0)$$

and

$$\Psi(v) = \int_0^v \psi(t)\, dt \quad (v \geqslant 0)$$

are called **mutually complementary Young functions**. By $M(u)$ we shall denote any one of these functions.

One says that a Young function $M(u)$ satisfies the Δ_2-condition for $u \geqslant u_0 \geqslant 0$, if there exists a constant $C > 0$ such that

$$M(2u) \leqslant CM(u) \quad \text{for } u \geqslant u_0.$$

Let B be a set of finite or infinite Lebesgue measure in an s-dimensional Euclidean space. By L_M we denote the class of real-valued functions $u(x)$, measurable on B, for which

$$\int_B M[|u(x)|]\, dx < \infty,$$

and by L^M, following [7], we denote the generalized Orlicz space, which is defined in exactly the same way as the ordinary Orlicz space.

In particular, if $\varphi(u) \equiv 1$ $(0 < u < \infty)$, then $L^\Phi = L^1$, $L^\Psi = L^\infty$. In general, if $\lim_{u\to\infty} \varphi(u) = l < \infty$ and $\operatorname{mes} B < \infty$, then L^Φ and L^Ψ contain the same functions as L^1 and, respectively, L^∞.

We note that the fundamental facts of the theory of Orlicz spaces remain valid also for the spaces generalized in the sense of Zaanen.

2. We proceed to present the results obtained by the author. Let L^M and L^{M_1} be arbitrary Orlicz spaces. Put

$$d = \sup\{u \in [0, \infty) : \ M(u) < \infty\}$$

and, for an arbitrary positive c, define the auxiliary functions

$$f_c(v) = \sup\{u \in [0, d) : \ M_1(uv) \geqslant cM(u)\},$$

$$F_c(v) = vf_c(v) \quad (0 \leqslant v \leqslant \infty).$$

We note the following properties of the function $f_c(v)$:

1°. $f_c(v)$ does not decrease with respect to v for fixed c, and

$$\lim_{v\to\infty} f_c(v) = d.$$

2°. $f_c(v)$ does not increase with respect to c for fixed v.

3°. $f_{kc}(kv) \geqslant f_c(v)$ for $k \geqslant 1$ and for all c and v.

4°. $M_1[F_c(v)] \geqslant cM[f_c(v)]$ for all c and v; if $M(d) = \infty$, then this inequality becomes an equality.

5°. If any one of the following relations:

$$f_c(v) = d$$

either for sufficiently large v, or for all v;

$$f_c(v) < d$$

either for sufficiently small v, or for all finite v;

$$f_c(v) = 0$$

for sufficiently small v,

holds for some $c = c_0$, then it is true for all c. This property follows from 2° and 3°.

3. Consider the multiplication operator H, $Hu = b(x)u(x)$, where $b(x)$ is a real-valued function measurable on B.

Theorem 1. *For the operator H to act from L^M into L^{M_1}, and also for it to be continuous from L^M into L^{M_1}, it is necessary and sufficient that, for some c and $\lambda > 0$,*

$$F_c(\lambda |b(x)|) \in L_{M_1}.$$

Sufficiency is proved directly. In the proof of necessity, four cases are considered (see 5°):

a) $f_c(v) = \infty$ for all $v > 0$;

b) $M_1(u) < \infty$ $(0 \le u < \infty)$ and $f_c(v) < \infty$ for all finite v;

c) $f_c(v) < \infty$ for sufficiently small v and $f_c(v) = \infty$ for sufficiently large v;

d) $M_1(u) = \infty$ for sufficiently large u.

Let us note that case d) overlaps with case a). In each case the proof uses the corresponding construction. In case a) it is proved that $b(x) = 0$ almost everywhere on B. In case b), assuming that $F_c(\lambda|b(x)|) \notin L_{M_1}$ for any c and $\lambda > 0$, we construct a sequence $\{B_n\}$ of pairwise disjoint subsets of the set B such that

$$\int_{B_n} M_1\left[F_n\left(\frac{1}{n}|b(x)|\right)\right] dx = \frac{1}{n}, \qquad n = 1, 2, \dots .$$

Next we set

$$u(x) = \begin{cases} f_n\left(\frac{1}{n}|b(x)|\right), & \text{for } x \in B_n, \quad n = 1, 2, \dots, \\ 0, & \text{for } x \in B \setminus \bigcup_{n=1}^{\infty} B_n. \end{cases}$$

Then it turns out that $u(x) \in L_M \subset L^M$, while $Hu \notin L^{M_1}$. In cases c) and d) it is first proved that $b(x) \in L^\infty$. Then the argument is carried out analogously to case b).

Corollary. If the operator H maps L^M into L^{M_1}, then it is continuous.

Let us point out some particular criteria for the action of the operator H from L^M into L^{M_1}.

Let $d < \infty$. Then, for H to act from L^M into L^{M_1}, it is sufficient, and if $\operatorname{mes} B < \infty$ also necessary, that $b(x) \in L^{M_1}$.

Let $f_c(v) = \infty$ for all $v > 0$. Then, for H to act from L^M into L^{M_1}, it is necessary and sufficient that $b(x) = 0$ almost everywhere on B.

Let $f_c(v) = 0$ for sufficiently small v (if $\operatorname{mes} B < \infty$, it suffices to assume that $f_c(v) < \infty$ for sufficiently small v). Then, for H to act from L^M into L^{M_1}, it is sufficient, and if $f_c(v) = \infty$ for sufficiently large v, also necessary, that $b(x) \in L^\infty$.

4. Finally, let us pass to the operator h, $hu = g(u(x), x)$, where the real-valued function $g(u, x)$ is continuous in u for almost every fixed $x \in B$ and is measurable on B in x for every fixed $u \in (-\infty, \infty)$.

Theorem 2. Suppose that $M(u)$ satisfies the Δ_2-condition for $u \geq 0$. Then, for the weak continuity of the operator h from L^M into L^{M_1}, the following condition is necessary and sufficient:

α) Almost for all $x \in B$, the function $g(u, x)$ has the form

$$g(u, x) = a(x) + b(x)u \qquad (-\infty < u < \infty),$$

where $a(x) \in L^{M_1}$, while $b(x)$ is such that $f_c(\lambda |b(x)|) \in L_M$ for some c and $\lambda > 0$.

Remark 1. If $\operatorname{mes} B < \infty$, then Theorem 2 remains valid under the assumption that $M(u)$ satisfies the Δ_2-condition for $u \geq u_0 > 0$ and the inequality $M(u) < \infty$ for $u \in (0, \infty)$.

The sufficiency of condition α) for the weak continuity of h from L^M into L^{M_1} follows from property 4° of the function $f_c(v)$ and Theorem 1, since under the hypotheses of Theorem 2 we have $M(d) = M(\infty) = \infty$.

Remark 2. The proof of sufficiency does not use the Δ_2-condition as such, but uses only the finiteness of the function $M(u)$. If, however, in condition α), instead of the summability of $f_c(\lambda |b(x)|)$ with the function $M(u)$

require that $F_c(\lambda |b(x)|) \in L_{M_1}$, then in proving sufficiency one may also dispense with the finiteness of $M(u)$.

For the proof of the necessity of condition α), one considers the function $G(u, x) = g(u, x) - g(0, x)$ and the set P of those $x \in B$ for which $G(u, x) \neq G(1, x)u$ for at least one $u \in (-\infty, \infty)$.

From the assumption that $\operatorname{mes} P > 0$, it follows that there exist two numbers $u' > 0$ and $u'' < 0$ such that the set of those $x \in B$ for which the inequality $u'G(u'', x) \neq u''G(u', x)$ holds has positive measure. From this set one selects a bounded closed subset e of positive measure on which the functions $G(u', x)$ and $G(u'', x)$ are continuous and the difference $u'G(u'', x) - u''G(u', x)$ preserves its sign. Then, in a definite way, one constructs a sequence of functions $\{u_n(x)\}$

(each of which vanishes on the set $B \setminus e$, and on the set e almost everywhere assumes only two values: u' and u'') such that $\int_D u_n(x)\,dx \to 0$ $(n \to \infty)$ for every measurable set $D \subset B$. Hence it follows that for every function $v(x) \in L^N$, where $N(v)$ is the Young function complementary to $M(u)$,

$$\lim_{n\to\infty} \int_B u_n(x)v(x)\,dx = 0.$$

And this means that the sequence $\{u_n(x)\}$ converges weakly in L^M to $u_0(x) \equiv 0$, since under the conditions of the theorem every linear functional in L^M has the form ($^{2-4}$)

$$lu = \int_B u(x)v(x)\,dx,$$

where $v(x) \in L^N$. On the other hand, from the construction of the set e and of the sequence $\{u_n(x)\}$ it follows that $\lim_{n\to\infty} \int_e G(u_n(x), x)\,dx \neq 0$, i.e. that the sequence $\{hu_n\}$ does not converge weakly in L^{M_1} to hu_0. Consequently, from the weak continuity of the operator h from L^M into L^{M_1} it follows that $\operatorname{mes} P = 0$.

To complete the proof it remains to apply Theorem 1 and property 4° of the function $f_c(v)$.

In the case where $L^M = L^p,\ L^{M_1} = L^{p_1}$ $(1 \le p < \infty,\ 1 \le p_1 \le \infty)$, Theorem 2 gives a necessary and sufficient condition for the weak continuity of the Nemytskii operator from L^p into L^{p_1}. In this case $a(x) \in L^{p_1}$; $b(x) \equiv 0$, if $p_1 > p$, and $b(x) \in L^r,\ r^{-1} = p_1^{-1} - p^{-1}$, if $p_1 \le p$.

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Note: Figure translations are in progress. See original paper for figures.