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Chr. Ya. Khristov

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Abstract

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Physics

Chr. Ya. Khristov

On the Diffusion of Charged Particles in a Homogeneous Electromagnetic Field

(Presented by Academician M. A. Leontovich, 4 VI 1957)

In recently published works ([1–3]) we considered the diffusion of molecules, neutrons, and avalanche particles. We wrote the equations and gave approximate expressions for the probabilities $V_{ij}(s, \mathbf{q}, \mathbf{u}, t, \mathbf{r}, \mathbf{v})\, d\mathbf{r}\, d\mathbf{v}$ that a particle of type i, which at time s had position $\mathbf{q}$ and velocity $\mathbf{u}$, creates at time t a particle of type j with radius vector lying between $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$, and with velocity between $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$. Using the notation and results of ([3]), we shall write the equation for the function V and give a method for the approximate solution of this equation when the particles whose diffusion we are investigating are charged and move in a homogeneous and time-independent medium situated in a homogeneous electromagnetic field. In doing so we shall assume that the particles are identical and that, in their collisions with the molecules of the medium, they may not only be scattered, but also be absorbed and generate new particles, while the motion of each particle between collisions is determined only by the prescribed external electromagnetic field, and not by the fields of the other particles. This process occurs, for example, in a weakly ionized plasma in a stationary state, where ions of a given type collide with one another, as well as with other ions and gas molecules, and in this process are not only scattered, but also recombine and generate new ions by impact ionization ([4]).

The function V satisfies an equation of type (17) from ([3]). However, in the present work this equation can be simplified somewhat, since the problem is more specific. Namely:

1. In view of the fact that, in the case considered, the avalanche consists of particles of a single type, we shall have $n = 1$, and the indices i, j drop out.

2. If we assume, without loss of generality, that the magnetic field $\mathbf{H}$ is directed along the x-axis, and the electric field $\mathbf{E}$ lies in the x, y-plane, then the function $\mathbf{F}$ in equation (1) from ([3]) takes the form $\frac{e}{m}\mathbf{E} + \frac{\mu e}{m}\mathbf{v} \times \mathbf{H}$, and integration of this equation gives, for $\vec{\varphi}^{\,s}$ and $\vec{\psi}^{\,s}$ (((2) from ([3]))),

$$\vec{\varphi}^{\,s}(t, \mathbf{r}, \mathbf{v}) = \mathbf{r} + \mathbf{h}_{s-t}(\mathbf{v}), \qquad \vec{\psi}^{\,s}(t, \mathbf{r}, \mathbf{v}) = \mathbf{k}_{s-t}(\mathbf{v}),$$

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where ([4])

$$\mathbf{h}_t(\mathbf{v}) = \mathbf{m}_t + \mathbf{v}\cdot\mu_t, \qquad \mathbf{k}_t\mathbf{v} = \mathbf{n}_t + \mathbf{v}\cdot\nu_t,$$

$$m_0 = \frac{1}{2}\chi_0 t^2, \qquad m_1 = \frac{\chi_1}{\omega^2}(1-\cos\omega t), \qquad m_2 = -\frac{\chi_1}{\omega}t + \frac{\chi_1}{\omega^2}\sin\omega t,$$

$$n_0 = \chi_0 t, \qquad n_1 = \frac{\chi_1}{\omega}\sin\omega t, \qquad n_2 = -\frac{\chi_1}{\omega}(1-\cos\omega t),$$

$$\mu_0 = t, \qquad \mu_1 = \frac{1}{\omega}\sin\omega t, \qquad \mu_2 = \frac{1}{\omega}(1-\cos\omega t),$$

$$\gamma_0 = 1, \qquad \gamma_1 = \cos\omega t, \qquad \gamma_2 = -\sin\omega t,$$

$$\chi_0 = \frac{e}{m}E_0, \qquad \chi_1 = \frac{e}{m}E_1, \qquad \omega = \frac{\mu e}{m}H \;\; * . \tag{4}$$

3. In view of the homogeneity of the medium we obtain

$$V(s,\mathbf{q},\mathbf{u},t,\mathbf{r},\mathbf{v}) = V(t-s,\mathbf{r}-\mathbf{q},\mathbf{u},\mathbf{v}), \qquad P(t,\mathbf{r},\mathbf{v},\mathbf{w}) = P(\mathbf{v},\mathbf{w}),$$

$$p(t,\mathbf{r},\mathbf{v}) = p(\mathbf{v}), \quad M(t,\mathbf{r},\mathbf{v},s) = \exp\left[-\left|\int_t^s p(\mathbf{k}_\tau(\mathbf{v}))d\tau\right|\right] = M(s-t,\mathbf{v}).$$

For V we obtain the integral equation

$$V(t,\mathbf{r},\mathbf{u},\mathbf{v}) = \delta(\mathbf{r}-\mathbf{h}_{-t}(\mathbf{v}))\delta(\mathbf{u}-\mathbf{k}_{-t}(\mathbf{v}))M(-t,\mathbf{v}) +$$

$$+\int_0^t\int V(t-\tau,\mathbf{r}-\mathbf{h}_{-\tau}(\mathbf{v}),\mathbf{u},\mathbf{w})P(\mathbf{w},\mathbf{k}_{-\tau})M(-\tau,\mathbf{v})\,d\tau\,d\mathbf{v}. \tag{1}$$

The approximate solution of this equation is found, in general terms, as in ([3]). Therefore we shall indicate only a certain difference in the conclusions and give the results directly. The solution is obtained by integrating (23) of ([3]). However, in order to find the initial approximation V_0 (24) of ([3]), we must generalize not the solution of the Fokker–Planck equation (250) of ( ) for free Brownian motion, given by Chandrasekhar by formula (178), as we did in ([1]-[3]), but the expression giving the analogous solution of the Fokker–Planck equation for Brownian motion in an electromagnetic field (( ), p. 249)

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$$\frac{\partial V}{\partial t} + \mathbf{v}\cdot\frac{\partial V}{\partial \mathbf{r}} + \frac{\partial}{\partial \mathbf{v}}\cdot\Big[\Big(\frac{e}{m}\mathbf{E} + \frac{\mu e}{m}\mathbf{v}\times\mathbf{H} - \beta\mathbf{v}\Big)V\Big] - \frac{\beta c^2}{2}\frac{\partial}{\partial \mathbf{v}}\cdot\frac{\partial W}{\partial \mathbf{v}} = 0. \quad (2)$$

First we consider the case when there is no electric field $\mathbf{E}$, and put

$$V_0(t,\mathbf{r},\mathbf{u},\mathbf{v}) = \frac{\beta^3}{\pi^3}\, g\,(1 + g'e^{-\lambda' t} + g''e^{-\lambda'' t} + \ldots)\sqrt{a^{*3}a^3\alpha_0^*\alpha_0\alpha_1^{*2}\alpha_1^2}\times \quad (3)$$
$$\times\exp\{-\lambda t - \alpha^* a(\mathbf{v} - \mathbf{u}\cdot\beta)^2 - (\beta\mathbf{r} - \mathbf{u}\cdot\gamma - \mathbf{v}\cdot\delta)\cdot\alpha^*\cdot(\beta\mathbf{r} - \tilde{\gamma}\cdot\mathbf{u} - \tilde{\delta}\cdot\mathbf{v})\}.$$

Here the scalar a and the tensors $\alpha, \beta, \gamma, \delta$ are functions of t, specified by

$$\frac{1}{a} = 1 - e^{-2\beta t}, \qquad \beta_0 = e^{-\beta t}, \qquad \beta_1 = \cos\omega t\, e^{-\beta t}, \qquad \beta_2 = \sin\omega t\, e^{-\beta t},$$

$$\delta_0 = \frac{1 - e^{-\beta t}}{1 + e^{-\beta t}}, \qquad \delta_1 = \frac{\beta^2}{\beta^2 + \omega^2}\left(\operatorname{cth}\beta t - \frac{\cos\omega t}{\operatorname{sh}\beta t}\right), \qquad \delta_2 = \frac{\beta^2}{\beta^2 + \omega^2}\left(\frac{\sin\omega t}{\operatorname{sh}\beta t} - \frac{\omega}{\beta}\right),$$

$$\frac{1}{\alpha_0} = \beta t - 2\frac{1 - e^{-\beta t}}{1 + e^{-\beta t}}, \qquad \frac{1}{\alpha_1} = \int_0^t (\delta_1^2 + \delta_2^2)\,\beta\, dt, \qquad \alpha_2 = 0, \quad (4)$$

$$\gamma_0 = \frac{1 - e^{-\beta t}}{1 + e^{-\beta t}}, \qquad \gamma_1 = 2\int_0^t \frac{\beta_1\delta_1 - \beta_2\delta_2}{1 - e^{-2\beta t}}\,\beta\, dt, \qquad \gamma_2 = 2\int_0^t \frac{\beta_1\delta_2 + \beta_2\delta_1}{1 - e^{-2\beta t}}\,\beta\, dt,$$

and the quantities

$$a^*,\ \alpha^*,\ \beta,\ \lambda,\ \lambda',\ \lambda_1'', \ldots,\ g,\ g',\ g'', \ldots \quad (5)$$

* We shall denote the components of a vector $\mathbf{l}$ by $l_0 = l_x,\ l_1 = l_y,\ l_2 = l_z$. For all tensors ε that we introduce, the relations $\varepsilon_{xy} = \varepsilon_{xz} = \varepsilon_{yx} = \varepsilon_{zx} = 0,\ \varepsilon_{yy} = \varepsilon_{zz},\ \varepsilon_{zy} = -\varepsilon_{yz}$ hold. Therefore each tensor ε is characterized by the quantities $\varepsilon_0 = \varepsilon_{xx}, \varepsilon_1 = \varepsilon_{yy}, \varepsilon_2 = \varepsilon_{zy}$. By $\tilde{\varepsilon}$ we shall denote the transposed tensor ε.

are functions of $\mathbf{u}$, but not of t, which should be determined so that V_0 (3) is the best possible approximation to V for large t. In this case the tensor α^* must be symmetric, so that $\alpha_2^* = 0$. A more general formulation, introducing factors independent of t before β, γ, δ, cannot be made, because this would violate the initial conditions at $t = 0$.

The solution of equation (2) is obtained from (3) for

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$$a^* = \frac{1}{c^2}, \qquad \alpha^* = \frac{1}{2c^2}, \qquad \lambda = 0, \qquad g = 1, \qquad g' = g'' = ... = 0. \quad (6)$$

To obtain the equation corresponding to equations (47) and (53) from (3), we denote here by $W^m(\mathbf{u}, \mathbf{v})\, d\mathbf{v}$ the probability that, among the particles of the m-th generation in the avalanche produced by one particle with initial velocity $\mathbf{u}$, there is one with initial velocity between $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$. Bearing in mind that for $\mathbf{E} = 0$ we have $\mathbf{m}_t = 0$, $\mathbf{n}_t = 0$, $M(t, v) = e^{-p(v)|t|}$, we obtain

$$W' = \delta(\mathbf{v} - \mathbf{u}), \qquad W^{m+1}(\mathbf{u}, \mathbf{v}) = \int W^m(\mathbf{u}, \mathbf{w})\, Q(\mathbf{w}, \mathbf{v})\, d\mathbf{w},$$

where

$$Q(\mathbf{u}, \mathbf{v})\, d\mathbf{v} = \frac{\displaystyle\int_0^\infty e^{-p(u)t} P(\mathbf{k}_t(\mathbf{u}), \mathbf{v})\, dt}{\displaystyle\int_0^\infty \int e^{-p(u)t} P(\mathbf{k}_t(\mathbf{u}), \mathbf{v})\, dt\, d\mathbf{v}}\, d\mathbf{v}$$

is the probability of finding a particle with initial velocity between $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$ among the particles obtained in the collision of a particle with initial velocity $\mathbf{u}$.

For the time-averaged displacement between two collisions of a particle with initial velocity $\mathbf{u}$, instead of $\mathbf{u}/p(u)$ (3) we obtain

$$\bar{\mathbf{r}}(\mathbf{u}) = \int_0^\infty \mathbf{h}_t(\mathbf{u}) e^{-p(u)t} p(u)\, dt.$$

Let us denote by

$$U(u) = \int_{-\pi^{3/2}}^{a^{*3/2}} \exp(-a^* v^2)\, \frac{P(\mathbf{v}, \mathbf{u})}{\displaystyle\int P(\mathbf{v}, \mathbf{u})\, d\mathbf{u}}.$$

the distribution function of particles with respect to velocities for large t.

Then for $\bar{t}$, $\bar{\mathbf{r}}^m$, $\overline{\mathbf{r}'\mathbf{r}'}$, $\overline{\mathbf{r}'\mathbf{r}^m} + \overline{\mathbf{r}^m\mathbf{r}'}^*$ instead of (50), (45), (51), and (52) from (3), we find

$$\bar{t} = \int \frac{U(u)}{p(u)}\, d\mathbf{u}, \qquad \bar{\mathbf{r}}^m(\mathbf{u}) = \int W^m(\mathbf{u}, \mathbf{v})\, \bar{\mathbf{r}}(\mathbf{v})\, d\mathbf{v},$$

$$\overline{\mathbf{r}'\mathbf{r}'} = 2 \int U(u)\, \bar{\mathbf{r}}(\mathbf{u}) \bar{\mathbf{r}}(\mathbf{u})\, d\mathbf{u},$$

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$$\overline{\mathbf{r}'\mathbf{r}^m} + \overline{\mathbf{r}^m\mathbf{r}'} = \iint U(u) W^m(\mathbf{u}, \mathbf{v}) \left[\mathbf{r}(\mathbf{u})\mathbf{r}(\mathbf{v}) + \mathbf{r}(\mathbf{v})\mathbf{r}(\mathbf{u})\right] d\mathbf{u}\, d\mathbf{v}.$$

With the aid of these formulas, instead of (47) and (53) from (3), for the equations expressing the equality of the mean displacement of particles and their mean dispersion, computed from (3) or by Yang' s method ([6]), we obtain

$$\frac{1}{\beta} \int U(u) \left|\mathbf{u} \cdot \gamma_{t=\infty}\right| \, d\mathbf{u} = \int U(u) \left|\mathbf{r}'(\mathbf{u}) + \mathbf{r}''(\mathbf{u}) + ...\right| \, d\mathbf{u}; \tag{7}$$

* The product of two vectors without a sign between them denotes their tensor product.

$$-\frac{1}{4\beta^2 a^* \cdot (ta)_{t=\infty}} = \frac{1}{6t} \left(\overline{r'r'} + \overline{r'r''} + \overline{r''r'} + \overline{r'r'''} + \overline{r'''r'} + \cdots\right). \tag{8}$$

For the equations expressing the condition that V_0 should satisfy (1) as well as possible for large t, instead of (36) and (37) from (3) we here find

$$1 - \int \frac{e(v)}{p(v) - \lambda} \, dv = 0; \tag{9}$$

$$\iint \left[\frac{3}{2} \frac{e(v)}{(p(v) - \lambda)^2} + \frac{\mathbf{v} \cdot \alpha^* \cdot \delta_{t=\infty} \cdot \tilde{\delta}_{t=\infty} \cdot (e(v) - \varepsilon(v)) \cdot \mathbf{v}}{p(v) - \lambda}\right] dv = 0, \tag{10}$$

where $e(v)$ and $\varepsilon(v)$ are given by the expressions

$$e(v) = \int U(w) P(\mathbf{w}, \mathbf{v}) \, d\mathbf{w}, \qquad v_0^2 \varepsilon_0(v) = \int w_0^2 U(w) P(\mathbf{w}, \mathbf{v}) \, d\mathbf{w},$$

$$(v_1^2 + v_2^2)\varepsilon_1(v) = \int (w_1^2 + w_2^2) U(w) P(\mathbf{w}, \mathbf{v}) \, d\mathbf{w}, \qquad \varepsilon_2(v) = 0.$$

Formulas (7), (8), (9), and (10) are sufficient for determining the quantities a^*, α^*, β, λ. It is clear that, as in (3), for them one obtains expressions that do not depend on $\mathbf{r}$ and $\mathbf{v}$, nor on $\mathbf{u}$, which simplifies their determination.

As for the quantities λ', λ'', ..., g, g', g'', ..., they are given by expressions (56) from (3), and if we restrict ourselves to two terms of the series in the bracket in (3), by expressions (57) from (3). Thus all quantities (5) have been found.

If the electric field $\mathbf{E}$ is not equal to zero, the solution of the Fokker–Planck equation (2) is given by formula (3) with the replacement

$$\mathbf{v}-\mathbf{u}\cdot\beta\to\mathbf{v}-\mathbf{u}\cdot\beta-\mathbf{b},\qquad \beta\mathbf{r}-\mathbf{u}\cdot\gamma-\mathbf{v}\cdot\delta\to\beta\mathbf{r}-\mathbf{u}\cdot\gamma-\mathbf{v}\cdot\delta-\mathbf{d},\quad(11)$$

where the vectors **b** and **d** are given by

$$b_0=\frac{\chi_0}{\beta}(1-e^{-\beta t}),\qquad b_1=\frac{\chi_1}{\beta^2+\omega^2}\left(\omega\sin\omega t\,e^{-\beta t}-\beta\cos\omega t\,e^{-\beta t}+\beta\right),$$

$$b_2=\frac{\chi_1}{\beta^2+\omega^2}\left(\omega\cos\omega t\,e^{-\beta t}+\beta\sin\omega t\,e^{-\beta t}-\omega\right),$$

$$d_0=\frac{\chi_0}{\beta}\left(\beta t-2\frac{1-e^{-\beta t}}{1+e^{-\beta t}}\right),\qquad d_1=\int_0^t\left[\frac{2\beta}{1-e^{-2\beta t}}(\hat{\delta}_1b_1+\hat{\delta}_2b_2)-(\hat{\delta}_1\gamma_1-\hat{\delta}_2\gamma_2)\right]dt,$$

$$d_2=\int_0^t\left[\frac{2\beta}{1-e^{-2\beta t}}(\hat{\delta}_1b_2-\hat{\delta}_2b_1)-(\hat{\delta}_1\gamma_2+\hat{\delta}_2\gamma_1)\right]dt.$$

In this case expressions (4) and (6) remain unchanged—the same as in the absence of an electric field. Therefore we can obtain an approximate solution V_0 of equation (1) in the presence of an electric field also from (3), with the replacement (11), preserving the method given above for finding the quantities (5). No new constants requiring determination appear when the electric field is included, so this method is applicable.

To obtain an idea of the accuracy of V_0, one may apply reasoning of the type given in ([1]).

Expression (3) and the indicated method for finding the quantities (5) may be of importance in considering not only stationary but also nonstationary processes in an ionized medium, since they make it possible to calculate the transport coefficients of this medium.

Sofia State University,
Moscow State University
named after M. V. Lomonosov

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[1] Chr. Christov, *Acta Phys. Hung.*, **7**, 51, 67 (1957).
[2] Khr. Ya. Khristov, DAN, **111**, 981 (1956).
[3] Khr. Ya. Khristov, ZhETF (in press).

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[4] S. Chapman, T. G. Cowling, *The Mathematical Theory of Non-uniform Gases*, Cambridge, 1953.
[5] S. Chandrasekhar, *Rev. Mod. Phys.*, **15**, 1 (1943).
[6] L. M. Yang, *Proc. Roy. Soc.*, **198**, 94 (1949).

Note: Figure translations are in progress. See original paper for figures.

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