MATHEMATICS

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1957

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Abstract

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MATHEMATICS

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ON CLASSES OF WELL-POSEDNESS FOR CERTAIN BOUNDARY-VALUE PROBLEMS

(Presented by Academician I. G. Petrovskii on 19 XII 1956)

In the present note we study the question of the well-posedness of certain boundary-value problems for partial differential equations. Equations of the type of the backward heat equation are considered, as well as certain equations of elliptic type. It is established that the boundary-value problems under investigation are well posed in the class of solutions uniformly bounded in the metric of a certain Hilbert space. Theorems on the well-posedness of the Cauchy problem in classes of solutions bounded in a certain sense, for other equations, were considered by Carleman (^{1}), M. M. Lavrent' ev (^{2}), E. M. Landis (^{3}), and John (^{4})*. The corresponding partial differential equations are treated by us as ordinary differential equations in a Hilbert space, which makes it possible to resolve the question of well-posedness by elementary means.

1. Let H be a Hilbert space; let $A(t)$, for each t in the segment $[0, T]$, be an unbounded operator acting in this space. We denote the domain of definition of the operator $A(t)$ by $D(A(t))$.

We shall consider solutions of the differential equation

$$dx/dt + A(t)x = 0 \tag{1}$$

under the initial conditions

$$x(0) = x_0. \tag{2}$$

Here by a solution of equation (1) we mean a function $x(t)$ with values in H, strongly continuous on the segment $[0, T]$, having a strong derivative at all interior points of this segment, and satisfying equation (1). It follows from the latter that $x(t) \in D(A(t))$ for $0 < t < T$.

Definition. We shall say that problem (1)–(2) is **well posed on the segment** $[0, T]$ **in the class of functions** $\mathfrak{M}$, if for every $\varepsilon > 0$ and $t \in [0, T]$ there exists $\delta = \delta(\varepsilon, t)$ such that, for every solution $x(t)$ of equation (1) belonging to the class $\mathfrak{M}$, from the inequality

$$\|x(0)\| \leq \delta \tag{3}$$

there follows the inequality

$$\|x(t)\| \leq \varepsilon. \tag{4}$$

The proof of the theorems on the well-posedness of problem (1)—(2) is based on the following fundamental theorem.

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* The author' s attention was drawn to these questions by a report of I. M. Gel' fand at the All-Union Conference on Functional Analysis and Its Applications ((^{7}), pp. 8-9).

Theorem 1. Let $A(t)$ be a self-adjoint operator whose domain of definition $D(A)$ does not depend on t, and suppose that on the domain of definition there exists a strong derivative dA/dt, with the inequality

$$\left(\frac{dA}{dt}x, x\right) \leq k(Ax, x) \qquad (x \in D(A),\ t \in (0, T)). \tag{5}$$

Then for every solution of equation (1) the inequality

$$\|x(t)\| \leq \|x(0)\|^{1-\alpha(t)} \|x(T)\|^{\alpha(t)}, \tag{6}$$

holds, where

$$\alpha(t) = \frac{e^{kt} - 1}{e^{kT} - 1}.$$

Proof. Consider the function

$$\varphi(t) = \ln(x(t), x(t)). \tag{7}$$

By a simple calculation we obtain

$$\frac{d\varphi}{dt} = -2\frac{(Ax, x)}{(x, x)}, \qquad \frac{d^2\varphi}{dt^2} = -2\frac{\left[\left(\frac{dA}{dt}x, x\right) - 2(Ax, Ax)\right](x, x) + 2(Ax, x)^2}{(x, x)^2}.$$

By the Cauchy-Schwarz inequality $(Ax, x)^2 \leq (Ax, Ax)(x, x)$, and therefore

$$\frac{d^2\varphi}{dt^2} \geq -2\frac{\left(\frac{dA}{dt}x, x\right)}{(x,x)} \geq -2k\frac{(Ax,x)}{(x,x)} = k\frac{d\varphi}{dt}. \tag{8}$$

Inequality (8) can be rewritten in the form

$$e^{-kt}\frac{d}{dt}\left(e^{-kt}\frac{d\varphi}{dt}\right) \geq 0. \tag{9}$$

If we make the substitution $r = (e^{kt}-1)/k$, $R = (e^{kT}-1)/k$, and $\varphi(t) = \psi(r)$, then it follows from inequality (9) that the function $\psi(r)$ is convex. Hence

$$\psi(r) \leq \left(1 - \frac{r}{R}\right)\psi(0) + \frac{r}{R}\psi(R). \tag{10}$$

Returning to the variable t and the function $\varphi(t)$, and using (7), from (10) we obtain (6). The theorem is proved.

Remark 1. In the case when the operator A is constant in (5), one may take $k = 0$, and inequality (6) takes the form

$$\|x(t)\| \leq \|x(0)\|^{1-t/T}\|x(T)\|^{t/T}.$$

The function $\ln(x,x)$ is then convex. This fact was known for the case when the solution has the form $x(t) = e^{At}x(0)$, where e^{At} is a semigroup of self-adjoint operators (^{5}).

Remark 2. Inequality (6) remains valid also in the case when $A(t)$ is a normal operator ($AA^* = A^*A$). In this case condition (5) should be replaced by the condition

$$\operatorname{Re}\left(\frac{dA}{dt}x, x\right) \leq k\operatorname{Re}(Ax,x). \tag{11}$$

Theorem 2. If in the differential equation (1) the operator $A(t)$ $(0 \leq t \leq T)$ is a normal operator satisfying condition (11), then problem (1)–(2) is well-posed on the segment $[0,T]$ in the class $\mathfrak{M}$ of functions uniformly bounded on $[0,T]$.

Indeed, let all functions of the class $\mathfrak{M}$ be bounded on $[0,T]$ by the constant M. If the solution $x(t) \in \mathfrak{M}$, then $\|x(T)\| \leqslant M$. Then from (6) it follows that

$$\|x(t)\| \leqslant \|x(0)\|^{1-\alpha(t)}M^{\alpha(t)}.$$

Choosing

$$\|x(0)\| \leqslant \delta = \left(\frac{\varepsilon}{M^{\alpha(t)}}\right)^{\frac{1}{1-\alpha(t)}},$$

we obtain $\|x(t)\| \leqslant \varepsilon$. The theorem is proved.

2. Let us now consider the differential equation

$$d^2x/dt^2 - Ax = 0, \tag{12}$$

where A is a constant unbounded normal operator whose spectrum does not contain and does not surround the point zero. The initial conditions will be prescribed in the form

$$x(0) = x_0, \qquad x'(0) = x'_0. \tag{13}$$

For the operator A one can always find a normal operator B such that $B^2 = A$. In this case there will exist a bounded inverse operator B^{-1}. If we now introduce the notation $y = B^{-1}dx/dt$, then equation (12) is replaced by the system

$$dx/dt - By = 0, \qquad dy/dt - Bx = 0.$$

From this system it is clear that the functions $x - y$ and $x + y$ satisfy equations of the form (1) with the operators B and $-B$, and, consequently, for them inequalities of the form (6) are valid. Combining them, we obtain

$$\|x(t)\| \leqslant \frac{1}{2}\{\|x(0) + B^{-1}x'(0)\|^{1-\alpha(t)} \cdot \|x(T) + B^{-1}x'(T)\|^{\alpha(t)} + \\ + \|x(0) - B^{-1}x'(0)\|^{1-\alpha(t)} \cdot \|x(T) - B^{-1}x'(T)\|^{\alpha(t)}\}. \tag{14}$$

Hence it follows:

Theorem 3. *If A is a constant normal operator whose spectrum does not contain and does not surround the point zero, then problem (1)–(2) is well-posed in the class $\mathfrak{M}$ of functions with derivatives with respect to t uniformly bounded on the segment $[0, T]$.*

Theorem 3 follows from inequality (14) in the same way as Theorem 2 follows from inequality (6).

Remark 3. From the well-posedness of problems (1)–(2) and (12)–(13) there follows the uniqueness of their solutions.

3. We give examples of applications of Theorems 2 and 3.

1) Let G be an n-dimensional domain. Denote by $L_2(G)$ the space of all functions square-summable over the domain G, with norm

$$\|u\|^2 = \int_G \cdots \int |u|^2 \, dx^*.$$

Let Λ be a differential operator of order $2m$, self-adjoint under certain homogeneous boundary conditions

$$\Gamma_1(u) = 0, \dots, \Gamma_{m-1}(u) = 0. \tag{15}$$

Let $\hat{G}$ be the cylindrical domain $G \times (0, T)$ in $(n+1)$-dimensional space. In this domain consider the following problem. It is required to find a solution $u(t, x)$ of the equation

$$\partial u/\partial t + \Lambda u = 0,$$

satisfying, for all $t > 0$, the boundary conditions (15) and, for $t = 0$, the initial condition

$$u(0, x) = f(x). \tag{16}$$

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* Here and below x is an n-dimensional vector, dx is the volume element.

It follows from Theorem 2 that the problem posed is well-posed in the class of solutions uniformly bounded with respect to t in the metric $L_2(G)$. Inequality (6) in this case takes the form

$$\int_G \cdots \int |u(t,x)|^2\,dx \le \left[\int_G \cdots \int |u(0,x)|^2\,dx\right]^{1-t/T} \left[\int_G \cdots \int |u(T,x)|^2\,dx\right]^{t/T}.$$

Let us emphasize that, in our considerations, the type of the operator Λ plays no role.

2) Consider the equation

$$\partial^2 u/\partial t^2 + \Lambda u = 0.$$

For it, one seeks in $\hat{G}$ solutions satisfying, in addition to conditions (15) and (16), also the condition

$$\frac{\partial u}{\partial t}(0, x) = f_1(x).$$

It follows from the theorem that the problem thus posed is well-posed in the class of solutions with derivative $\partial u/\partial t$ uniformly bounded with respect to t in the metric $L_2(G)$, if the spectrum of the operator Λ does not contain the point zero.

3) Consider the equation with a self-adjoint operator of second order with coefficients depending on t:

$$\frac{\partial u}{\partial t} + \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} a_{ij}(t, x) \frac{\partial u}{\partial x_i} = 0. \tag{17}$$

The equation is solved under the boundary condition

$$u = 0 \tag{18}$$

on the boundary of the domain and under the initial condition $u(0, x) = f(x)$.

Let the coefficients $a_{ij}(t, x)$ be continuous in the closed domain and have continuous partial derivatives with respect to t there. Finally, suppose that for all (t, x) from $\hat{G}$ and for any real numbers $\xi_1, \ldots, \xi_n$ one has

$$\sum \frac{\partial a_{ij}(t, x)}{dt} \xi_i \xi_j \geq k \sum a_{ij}(t, x) \xi_i \xi_j, \tag{19}$$

where k is some real number.

Under these conditions, the boundary-value problem under consideration is well-posed in the class of solutions uniformly bounded with respect to t in $L_2(G)$.

Note that inequality (19) is satisfied for some k if: a) the form

$$\sum a_{ij} \xi_i \xi_j$$

preserves a constant sign for all (t, x) from the closed domain $\hat{G}$; b) if the form

$$\sum \frac{\partial a_{ij}}{\partial t} \xi_i \xi_j$$

is nonnegative for all (t, x); c) if the form

$$\sum a_{ij} \xi_i \xi_j$$

is sign-definite for all (t, x) except $t = 0$, and its derivative with respect to t is positive at $t = 0$ and all x.

In conclusion, let us note that all proofs can be carried out within the framework of classical theory.

Received
18 XII 1956

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Note: Figure translations are in progress. See original paper for figures.

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