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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

MATHEMATICAL PHYSICS

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DISTRIBUTION OF ELECTRICITY ON A THIN PARABOLOIDAL SEGMENT

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1. The problem of the distribution of electricity on a conductor having the form of a thin open surface presents considerable difficulties, and its solution is known only for a few special cases.

Apart from the classical results concerning the distribution of electricity on the surface of a circular and an elliptic plate and of a thin spherical segment (^{1}), a solution of the problem under consideration has been found for a plane ring ($^{2-3}$) and for a thin spherical surface with two symmetrically situated circular cutouts (^{4}). It should be noted that in the last two cases the solution has a complicated form and is connected with the use of insufficiently studied special functions.

Fig. 1

In the present paper, on the basis of a new approach to the problem, an exact solution is given for the problem of the distribution of electricity on a thin paraboloidal segment. It is shown that the density of the charge distribution over the surface of the conductor and its capacitance can be expressed in quadratures through an auxiliary function which is the solution of a one-dimensional Fredholm integral equation with a continuous kernel.

For the case when the surface has small curvature, a simple formula is obtained for the capacitance of the segment in the form of a series in powers of a small parameter.

2. Let the equation of the surface of the paraboloidal segment (S) be

$$\frac{z}{h} = \left(\frac{r}{R}\right)^2 - \left(\frac{R}{2h}\right)^2 \quad (0 \le r \le R),$$

where the quantities R and h have the meanings indicated in Fig. 1.

The problem of the free distribution of electricity reduces to determining the potential $u = u(r, z)$, satisfying Laplace' s equation $\Delta u = 0$ and the boundary conditions $u|_S = V,\ u|_\infty = 0$.

We introduce a system of paraboloidal coordinates (α, β), connected with the cylindrical coordinates (r, z) by the relations

$$r = \alpha\beta, \qquad z = \frac{1}{2}(\alpha^2 - \beta^2) \quad (0 \le \alpha < \infty,\ 0 \le \beta < \infty).$$

The equation of the surface (S) in the new coordinates takes the form

$$\beta = \beta_0 = \frac{R}{\sqrt{2h}}, \quad 0 \le \alpha \le a = \sqrt{2h}.$$

We shall seek the solution of the problem in the form

$$u = \int_0^\infty A(\lambda)\frac{I_0(\lambda\beta)}{I_0(\lambda\beta_0)} J_0(\lambda\alpha)\,d\lambda \qquad (0 \le \beta < \beta_0),$$

$$u = \int_0^\infty A(\lambda)\frac{K_0(\lambda\beta)}{K_0(\lambda\beta_0)} J_0(\lambda\alpha)\,d\lambda \qquad (\beta_0 < \beta < \infty), \tag{1}$$

where $A(\lambda)$ is an unknown function; $J_0(x)$, $I_0(x)$, and $K_0(x)$ are cylindrical functions.

The function u, defined by the equalities (1), formally satisfies Laplace' s equation and is continuous throughout space, including the surface $\beta = \beta_0$. From the boundary condition $u|_S = V$ and the condition of continuity of the normal derivative on the remaining part of the surface $\beta = \beta_0$, one obtains, for determining $A(\lambda)$, the paired integral equations:

$$\int_0^\infty A(\lambda) J_0(\lambda\alpha)\,d\lambda = V \qquad (0 \le \alpha \le a),$$

$$\int_0^\infty A(\lambda)\{2\beta_0 I_0(\lambda\beta_0)K_0(\lambda\beta_0)\}^{-1} J_0(\lambda\alpha)\,d\lambda = 0 \qquad (a < \alpha < \infty). \tag{2}$$

We shall seek the solution of these equations in the form

$$A(\lambda) = 2\lambda\beta_0 I_0(\lambda\beta_0)K_0(\lambda\beta_0)\int_0^a \varphi(t)\cos\lambda t\,dt, \tag{3}$$

where $\varphi(t)$ is a function whose derivative is continuous in the interval $(0, a)$. If (3) is transformed by integration by parts and the formula

$$\int_0^\infty J_0(\lambda\alpha)\sin\lambda t\,dt = 0 \qquad (0 \le t \le a < \alpha), \tag{4}$$

is used, it is easy to verify that the second of equations (2) is satisfied identically. Substitution of (3) into the first of the equations under consideration gives

$$\int_0^\infty [1 - g(\lambda)] J_0(\lambda\alpha)\,d\lambda \int_0^a \varphi(t)\cos\lambda t\,dt = V \qquad (0 \le \alpha \le a), \tag{5}$$

where

$$g(\lambda) = 1 - 2\lambda\beta_0 I_0(\lambda\beta_0) K_0(\lambda\beta_0).$$

Taking into account the relations*

$$\int_0^\infty J_0(\lambda\alpha)\cos\lambda t\,d\lambda = \begin{cases} (\alpha^2 - t^2)^{-1/2}, & 0 \le t < \alpha, \\ 0, & t > \alpha, \end{cases} \tag{6}$$

$$\int_0^\infty g(\lambda) J_0(\lambda\alpha)\cos\lambda t\,d\lambda = \frac{1}{\pi}\int_0^{\pi/2} [G(t - \alpha\sin\theta) + G(t + \alpha\sin\theta)]\,d\theta,$$

where $G(x)$ is the Fourier cosine transform of $g(\lambda)$

$$G(x) = \int_0^\infty g(\lambda)\cos\lambda x\,d\lambda, \tag{7}$$

we can write equation (5) in the form

$$\int_0^\alpha \frac{\varphi(t)\,dt}{\sqrt{\alpha^2 - t^2}} - \frac{1}{\pi}\int_0^\alpha \varphi(t)\,dt \int_0^{\pi/2} \{G(t - \alpha\sin\theta) + G(t + \alpha\sin\theta)\}\,d\theta = V \tag{8}$$

$$(0 \le \alpha \le a)$$

or, if in the first integral we put $t = \alpha\sin\theta$,

$$\int_0^{\pi/2} \left\{ \varphi(\alpha\sin\theta) - \frac{1}{\pi}\int_0^a \varphi(t)[G(t - \alpha\sin\theta) + G(t + \alpha\sin\theta)]\,dt \right\} d\theta = V \tag{9}$$

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* The second of these relations is obtained by replacing the Bessel function by its representation in terms of t

$$(0 \le \alpha \le a).$$

The last equality will be satisfied if $\varphi(t)$ is a solution of the Fredholm integral equation

$$\varphi(x) - \frac{1}{\pi}\int_0^a \varphi(t)[G(t-x) + G(t+x)]\,dt = \frac{2V}{\pi}. \tag{10}$$

The function $G(x)$ can be expressed in terms of complete elliptic integrals of the first and second kinds

$$G(x) = \frac{1-k^2}{2\beta_0 k}\,[K(k) - E(k)] \tag{11}$$

with modulus

$$k = \frac{|x|}{\sqrt{x^2 + 4\beta_0^2}}.$$

In what follows it is convenient to pass to dimensionless quantities by putting

$$x = a\xi, \qquad t = a\tau, \qquad \varphi(x) = \frac{2V}{\pi}\psi(\xi), \qquad \varphi(t) = \frac{2V}{\pi}\psi(\tau), \qquad \frac{h}{R} = \rho. \tag{12}$$

Equation (10) then takes the form

$$\psi(\xi) - \frac{\rho}{\pi}\int_0^1 \psi(\tau)\left\{\frac{1-\mu^2}{\mu}[K(\mu) - E(\mu)] + \frac{1-\nu^2}{\nu}[K(\nu) - E(\nu)]\right\} d\tau = 1 \tag{13}$$

$$(0 \le \xi \le 1),$$

where

$$\mu = \frac{\rho|\tau - \xi|}{\sqrt{1 + \rho^2(\tau-\xi)^2}}, \qquad \nu = \frac{\rho(\tau+\xi)}{\sqrt{1+\rho^2(\tau+\xi)^2}}.$$

Assuming that ψ is known, the potential distribution can be found from formula (1), after which the values of the surface density, the total charge, and the capacitance of the conductor are easily determined.

The simplest result is obtained for the capacitance of the segment, namely

$$C = \frac{2R}{\pi} \int_0^1 \psi(\tau)\, d\tau. \tag{14}$$

The actual construction of the function ψ is carried out by solving equation (13) by numerical methods. For small values of the parameter ρ, the solution of the equation under consideration can be found in the form of an expansion in powers of this parameter, which leads to the following expression for the capacitance of the segment:

$$C = \frac{2R}{\pi}\left[1 + \frac{1}{12}(2\rho)^2 - \frac{1}{48}(2\rho)^4 + \frac{31}{3360}(2\rho)^6 - \frac{1829}{128 \cdot 2835}(2\rho)^8 + \frac{99149}{1024 \cdot 31185}(2\rho)^{10} - ...\right] \quad (0 \leq \rho \leq {}^1/_2). \tag{15}$$

As numerical calculations show, this formula gives the value of the required capacitance with an accuracy up to 0.1%.

In conclusion, we note that the proposed method of solving the electrostatic problem is readily extended to the case when a conductor having the shape of a paraboloidal segment is placed in an arbitrary external field possessing axial symmetry.

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Note: Figure translations are in progress. See original paper for figures.

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