MATHEMATICS

G. I. NATANSON

1957

SovietRxiv

Abstract

Full Text

MATHEMATICS

G. I. NATANSON

ON THE THEORY OF APPROXIMATION OF FUNCTIONS BY LINEAR COMBINATIONS OF EIGENFUNCTIONS OF THE STURM-LIOUVILLE PROBLEM

(Presented by Academician V. I. Smirnov on 10 XII 1956)

As is known ([1]), the Sturm-Liouville problem

$$U''(x) + [\lambda - B(x)]U(x) = 0, \tag{1}$$

$$U'(0) - hU(0) = 0, \qquad U'(\pi) + HU(\pi) = 0, \tag{2}$$

where the function $B(x)$ is given and continuous on $[0, \pi]$, and h and H are real numbers (not necessarily positive), has an infinite sequence of simple eigenvalues $\lambda_0, \lambda_1, \lambda_2, \ldots$, and eigenfunctions corresponding to these numbers $U_0(x), U_1(x), U_2(x), \ldots$. It is also known ([1]) that the functions $U_n(x)$ are represented uniformly on $[0, \pi]$ by the asymptotic formula

$$U_n(x) = \cos nx + O(n^{-1}).$$

It is therefore natural to expect that the theory of approximation of functions by linear combinations of Sturm-Liouville functions has much in common with the theory of approximation of functions by trigonometric polynomials.

In this direction, interesting and important results have been obtained by various authors. In particular, the following theorem holds.

Theorem ([2]). *Let the function $B(x)$ in the Sturm-Liouville equation (1) have on $[0, \pi]$ a derivative of order $r - 1$, continuous and of bounded variation, and let the function $f(x)$ be such that $f^{(r)}(x) \in \operatorname{Lip} 1$ and*

$$f^{(i)}(0) = f^{(i)}(\pi) = 0 \quad \left(i = 0, 1, 2, \ldots, 2\left[\frac{r+1}{2}\right] - 1\right).$$

Then for every natural n there exists a linear combination $\Phi_n(x)$ of the functions $U_k(x)$ $(k = 0, 1, 2, \ldots, n)$ such that for all $x \in [0, \pi]$

$$|f(x) - \Phi_n(x)| < \frac{A}{n^{r+1}},$$

where the constant A does not depend on n.

We also note that, for linear combinations $\Phi_n(x)$ of the Sturm–Liouville functions $U_k(x)$ ($k = 0, 1, 2, \ldots, n$), Carlson ([3]) proved an analogue of S. N. Bernstein's inequality for the derivative of a trigonometric polynomial.

In the present note some new results are formulated in the theory of approximation of functions by linear combinations of Sturm–Liouville functions. We shall use the following notation:

$$\|f\| = \max |f(x)|, \qquad 0 \le x \le \pi;$$

$$\omega(f, \delta) = \sup_{|x-y| \le \delta} \{|f(x) - f(y)|\};$$

$$E_n^{SL}(f) = \inf_{c_k} \left\{ \max_{x \in [0,\pi]} \left| f(x) - \sum_{k=0}^{n} c_{kU} k(x) \right| \right\}.$$

Let us note that the best approximation $E_n^{SL}(f)$ is attained, i.e. there exists a linear aggregate

$$\Phi_n(x) = \sum_{k=0}^{n} c_k U_k(x)$$

such that

$$\max_{x \in [0,\pi]} |f(x) - \Phi_n(x)| = E_n^{SL}(f).$$

Theorem 1*. Let the function $B(x)$ in (1) be continuous. Then for every function $f(x)$ continuous on $[0, \pi]$ we have

$$E_n^{SL}(f) = O(1) \left[\omega\left(f, \frac{1}{n}\right) + \frac{1}{n} \|f\| \right],$$

where for the quantity $O(1)$ there is an estimate that depends neither on n nor on $f(x)$.

Theorem 2. Let the function $B(x)$ in (1) have $r - 1$ derivatives of bounded variation (in the case $r = 1$ the continuity of $B^{(0)}(x) = B(x)$ is additionally assumed). If $f(x)$ is given on $[0, \pi]$, has r derivatives on this interval, with $f^{(r)}(x)$ continuous, and the functions

$$L^{[i]} f \quad \left(i = 0, 1, 2, \ldots, \left[\frac{r-1}{2} \right] \right),$$

where

$$L^{(0)}f = f(x), \qquad L^{[1]}f = f''(x) - B(x)f(x), \qquad L^{[i]}f = L^{[1]}(L^{[i-1]}f),$$

satisfy the boundary conditions (2), then

$$E_n^{SL}(f) = O(n^{-r})\left[\omega\left(f^{(r)}, \frac{1}{n}\right) + \frac{1}{n}\sum_{m=0}^{r}\|f^{(m)}\|\right].$$

The boundary conditions imposed on the function $f(x)$ (their number is half the number of conditions in the cited theorem of Jackson) are analogous to the conditions considered by McEwen ([4,5]) and by Galbraith and Varshavsky ([6]).

Theorem 3. Let $\omega(\delta)$ be a function satisfying the following conditions:

a) $\omega(\delta)$ is continuous for $0 < \delta < +\infty$;

b) $0 < \omega(\delta') \leq \omega(\delta'')$ for $0 < \delta' < \delta''$;

c) $\omega(0) = 0$;

d) there exists a $K > 1$ such that

$$\overline{\lim}_{\delta\to 0+}\frac{\omega(K\delta)}{\omega(\delta)} < K.$$

Then, if

$$E_n^{SL}(f) = O\left(\omega\left(\frac{1}{n}\right)\right),$$

and only continuity is required of the function $B(x)$ in (1), then

$$\omega(f,\delta) = O(\omega(\delta)).$$

Theorem 4. Suppose that the function $B(x)$ in (1) has $r-1$ derivatives, with $B^{(r-1)}(x)$ bounded on $[0,\pi]$ (in the case $r = 1$ the function $B^{(0)}(x) = B(x)$ is assumed continuous). Suppose further that the function $\omega(\delta)$ satisfies conditions a), b), c) of Theorem 3 and, in addition, the condition:

d′) there exists a constant K such that

$$1 < \underline{\lim}_{\delta\to 0+}\frac{\omega(K\delta)}{\omega(\delta)} \leq \overline{\lim}_{\delta\to 0+}\frac{\omega(K\delta)}{\omega(\delta)} < K.$$

---

* Strictly speaking, this theorem is not entirely new, since it is implicitly contained in ([2]).

If

$$E_n^{SL}(f) = O\left(\frac{1}{n^2}\,\omega\left(\frac{1}{n}\right)\right),$$

then:

α) there exist continuous $f'(x)$, $f''(x)$, ..., $f^{(r)}(x)$;

β) $\omega(f^{(r)}, \delta) = O(\omega(\delta))$;

γ) the functions introduced above $L^{[i]}f$ $\left(i = 0, 1, 2, \dots, \left[\frac{r-1}{2}\right]\right)$ satisfy the boundary conditions (2).

Theorems 3 and 4 are analogues of the corresponding theorems of S. M. Lozinskii ([7]).

Theorem 5. Let the function $B(x)$ in (1) be continuous and of bounded variation on $[0, \pi]$. In order that the best approximation $E_n^{SL}(f)$ of a continuous function $f(x)$ on $[0, \pi]$ satisfy the inequality

$$E_n^{SL}(f) < \frac{A}{n},$$

where A does not depend on n, it is necessary and sufficient that the function $\bar{f}(x)$—the even 2π-periodic extension of $f(x)$—be quasi-smooth on $(-\infty, +\infty)$, i.e. that there exist a constant M such that for all x and $h > 0$

$$|\bar{f}(x+h) - 2\bar{f}(x) + \bar{f}(x-h)| < Mh.$$

Remark. In the formulation of Theorem 5, the condition of quasi-smoothness of $\bar{f}(x)$ on the whole axis cannot be replaced by the condition of quasi-smoothness of $f(x)$ on the interval $[0, \pi]$.

Indeed, the function

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k} \qquad (x \in [0, \pi])$$

will be quasi-smooth on $[0, \pi]$, whereas its even 2π-periodic extension is not quasi-smooth on $(-\infty, +\infty)$.

We give one more result concerning interpolation by means of Sturm–Liouville functions.

The basic property of the Lagrange interpolation polynomial

$$L_n[f;x] = \sum_{k=1}^{n} \frac{\omega(x)}{(x-x_k)\omega'(x_k)}\, f(x_k) \qquad \left(\omega(x) = \prod_{k=1}^{n}(x-x_k)\right),$$

namely its coincidence with $f(x)$ at $x = x_k$, is preserved if, instead of $\omega(x)$, one takes any differentiable function having simple zeros at the points x_k. In particular, one may set* $\omega(x) = U_n(x)$.

Theorem 6. Let the function $B(x)$ in (1) be continuous and of bounded variation on $[0,\pi]$. Put

$$L_n^{SL}[f;x] = \sum_{k=1}^{n} \frac{U_n(x)}{(x-x_k)U_n'(x_k)}\, f(x_k),$$

where $x_k = x_k^{(n)}$ are the zeros of $U_n(x)$. Then for any $a \in (0,\pi/2)$, uniformly on $[a,\pi-a]$, we have

$$f(x) - L_n^{SL}[f;x] = O(\ln n)\left[\omega\left(f,\frac{1}{n}\right) + \frac{1}{n}\|f\|\right].$$

* $U_n(x)$ has n simple zeros in $(0,\pi)$ ([1]).

Remark 1. If $B(x)$ is only continuous on $[0,\pi]$, then on $[a,\pi-a]$

$$f(x) - L_n^{SL}[f;x] = O(\ln n)\,\omega\left(f,\frac{1}{n}\right) = O(n^{-1/2})\|f\|.$$

Remark 2. Even for functions with good structural properties, the constructed process may diverge at $x=0$ and $x=\pi$.

Leningrad State Pedagogical Institute
named after A. I. Herzen

Received
4 XII 1956

REFERENCES

[1] B. M. Levitan, *Expansion in eigenfunctions*, 1950, pp. 11–14.
[2] D. Jackson, Trans. Am. Math. Soc., **15**, 439 (1914).
[3] E. Carlson, Trans. Am. Math. Soc., **26**, 230 (1924).
[4] W. H. McEwen, Bull. Am. Math. Soc., **45**, 576 (1939).
[5] W. H. McEwen, Am. Math. J., **63**, 29 (1941).
[6] A. S. Galbraith, S. E. Warschawski, Duke Math. J., **6**, 318 (1940).
[7] S. M. Lozinskii, DAN, **83**, 645 (1952).