ON THE REDUCTION OF QUASIUNITARY OPERATORS TO TRIANGULAR FORM

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Abstract

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MATHEMATICS

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ON THE REDUCTION OF QUASIUNITARY OPERATORS TO TRIANGULAR FORM

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1. It is known that every matrix of finite order can be reduced, by means of a unitary transformation, to triangular form. In the paper ([1]) an analogous question was solved for operators of class $(i\Omega)$. In the present note the problem of reducing to triangular form quasiunitary operators defined in a Hilbert space H is solved.

Definition. A linear operator T, defined in a Hilbert space H, is called quasiunitary if the operators $I - T^*T$ and $I - TT^*$ are finite-dimensional.

Here we shall consider quasiunitary operators T satisfying the conditions:

1) $\mathrm{Dim}(I - T^*T) = \mathrm{Dim}(I - TT^*)$;
2) there exists at least one point ζ_0, $|\zeta_0| < 1$, regular for the resolvent of the operator T.

Denoting $D_T = (I - T^*T)H$; $D'_T = (I - TT^*)$, we shall call the dimension of D_T and D'_T the **rank of nonunitarity** of the operator T, and the pair of numbers (p, q), where p is the number of positive and q the number of negative squares of the form $((I - T^*T)f, f)$ $(f \in H)$, the **signature of nonunitarity** of the operator T.

The subspace $G_T = H \ominus D_T$ is the largest subspace on which $T^* = T^{-1}$.

The maximal subspace $\mathfrak{M}_T \subseteq G_T$ invariant with respect to T will be called the **additional component** of the operator T. If $\mathfrak{M}_T = 0$, then the quasiunitary operator will be called **simple**.

Without loss of generality, one may assume that the point $\zeta = 0$ does not belong to the spectrum of the operator T.

It is not hard to show that the additional component coincides with the orthogonal complement of the linear closed span of $T^{*k}D_T$ $(k = 0, \pm 1, \pm 2, ...)$.

2. The operator $|I - TT^*|^{-1/2}$ has meaning on D'_T; therefore, in view of the relations

$$(T - \zeta I)(I - \zeta T^*)^{-1}|I - T^*T|^{1/2} = [T - \zeta(I - TT^*)(I - \zeta T^*)^{-1}]\,|I - T^*T|^{1/2},$$

$$TD_T \subseteq D'_T,$$

the operator function

$$W(\zeta) = |I - TT^*|^{1/2}(T - \zeta I)(I - \zeta T^*)^{-1}|I - T^*T|^{1/2}$$

has meaning for those ζ for which $\overline{\zeta}^{-1}$ is a regular point of the operator T.

$W(\zeta)$ maps D_T into D'_T ([4]).

Considering orthonormal bases $\{e_k\}$, $\{e'_k\}$ $(k = 1, 2, \dots, r)$ in D_T and D'_T, respectively, we introduce the matrix function

$$w_T(\zeta) = \left\|(W(\zeta)e_k, e'_j)\right\|_{k,j=1}^{r},$$

which we shall call the **normalized characteristic matrix-function** of the quasiunitary operator T. It is not difficult to show that this definition is equivalent to the definition of the analogous function in paper ([2]); therefore all assertions of that paper are valid for $w_T(\zeta)$.

Since $w_T(\zeta)$ satisfies all the conditions of Potapov' s main theorem ([3]), it can be represented in the following equivalent form:

$$w_T(\zeta) = \prod_{k=1}^{\infty} \mathfrak{U}_k \left\{ t(k) - \zeta p(k) \left[e^{i\varphi(k)} - \zeta t(k) \right]^{-1} p(k) J \right\} \mathfrak{U}_k^{-1} \times$$

$$\times \int_0^l \exp\left[\frac{\zeta + e^{i\varphi(x)}}{\zeta - e^{i\varphi(x)}} p^2(x) J \, dx \right] \mathfrak{B}, \tag{1}$$

where $t(k)$ is a sequence of positive diagonal matrices having one and only one eigenvalue different from unity, and

$$\prod_{k=1}^{\infty} \rho_k$$

converges ($\rho_k \neq 1$ are the eigenvalues of the matrices $t(k)$); $p^2(k) = |I - t^2(k)|$; $\{\mathfrak{U}_k\}$ is a sequence of J-unitary matrices such that

$$\prod_{k=1}^{\infty} \mathfrak{U}_k t(k) \mathfrak{U}_k^{-1}$$

converges. $p(x)$ is a Hermitian nonnegative matrix-function satisfying the condition $\operatorname{Sp} p^2(x) = 1$ $(0 \le x \le l)$; $\varphi(k)$ and $\varphi(x)$ are nondecreasing and bounded scalar functions, with $\varphi(x + 0) = \varphi(x)$.

3. In this section we construct a triangular model T of a quasiunitary operator with given nonunitarity rank r and given signature (p, q).

Let $\mathcal{H}_r$ be a Euclidean r-dimensional space. Consider the spaces $\mathbf{H}_1$ and $\mathbf{H}_2$ of all vector-functions $f(k)$ $(k = 1, 2, ...)$ and $f(x)$ $(0 \leq x \leq l)$ of discrete and continuous arguments k and x, respectively, whose values belong to $\mathcal{H}_r$. Form the direct sum $\mathbf{H} = \mathbf{H}_1 \dotplus \mathbf{H}_2$. The space $\mathbf{H}$ consists of all pairs of the form $f = \{f(k), f(x)\}$, where $f(k) \in \mathbf{H}_1$, $f(x) \in \mathbf{H}_2$.

We define the scalar product of vectors in $\mathbf{H}$ by the equality

$$(f, g) = \sum_{k=1}^{\infty} f(k) g^*(k) + \int_0^l f(x) g^*(x)\, dx, \qquad f, g \in \mathbf{H};$$

fg^* is the scalar product of vectors in $\mathcal{H}_r$.

Let $\{t(k)\}$, $\{\mathfrak{U}_k\}$, $\{p(k)\}$, $p(x)$, $\varphi(k)$, $\varphi(x)$ satisfy the conditions of the preceding section. Define in $\mathbf{H}$ the operator Tf by the equalities

$$Tf(k) = f(k) t(k) e^{i\varphi(k)} - \sum_{j=k+1}^{\infty} f(j) p(j) \mathfrak{U}_j \pi^*(j-1) \pi^{*-1}(k) \mathfrak{U}_k^{*-1} J p(k) e^{i\varphi(k)} - \int_0^l f(t) \sqrt{2}\, p(t) \pi^*(t) \pi^{*-1}(k) \mathfrak{U}_k^{*-1} J p(k) \tag{2}$$

$$Tf(x) = f(x) e^{i\varphi(x)} - 2 \int_x^l f(t) p(t) \pi^*(t) \pi^{*-1}(x) J p(x) e^{i\varphi(x)}\, dt, \tag{3}$$

where

$$\pi(k) = \sum_{j=1}^{k} \mathfrak{U}_j t(j) \mathfrak{U}_j^{-1}, \qquad \pi(x) = \pi(k)\big|_{k=\infty} \int_0^x e^{-p^2(t) J\, dt}, \qquad J = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

Theorem 1. *The operator T has the following properties:*

1) T is a quasiunitary operator of rank r with signature (p, q), where p, q are equal, respectively, to the number of $+1$ and -1 entries of the matrix J.

2) The spectrum of the operator T, not lying on the circle $|\zeta| = 1$, consists of the set of points of the form $\zeta_k = \rho_k e^{i\varphi(k)}$ $(k = 1, 2, ...)$, where ρ_k are the eigenvalues of the matrices $t(k)$, different from unity. The points ζ_k are poles of the resolvent.

3) The spectrum of the operator T lying on the circle $|\zeta| = 1$ coincides with the set $\mathfrak{R}$ of values taken by the function $e^{i\varphi(x)}$ $(0 \leq x \leq l)$. These values may be essentially singular points of the resolvent.

4) We represent the adjoint operator T^*f in the form

$$T^*f(k) = f(k)t(k)e^{-i\varphi(k)} - \sum_{j=1}^{k-1} f(j)e^{-i\varphi(j)}p(j)J\mathfrak{U}_j^{-1}\pi^{-1}(j)\pi(k-1)\mathfrak{U}_k p(k); \quad (4)$$

$$T^*f(x) = f(x)e^{-i\varphi(x)} - 2\int_0^x f(t)e^{-i\varphi(t)}p(t)J\varkappa^{-1}(t)\pi(x)p(x)\,dt$$
$$- \sum_{j=1}^{\infty} f(j)e^{-i\varphi(j)}p(j)J\mathfrak{U}_j^{-1}\pi^{-1}(j)\pi(x)\sqrt{2}\,p(x). \quad (5)$$

5) The normalized characteristic matrix-function of the operator T is determined by formula (1).

An operator T satisfying the conditions of Theorem 1 will be called a triangular model of a quasiunitary operator.

Theorem 2. For every quasiunitary operator T of rank r with signature (p,q) satisfying condition 2) of item 1, one can construct a triangular model of the form (2), (3) and a unitary transformation U, possessing the following properties:

1) The operator U maps $H \dot{-} \mathfrak{M}_T$ one-to-one onto the space $H \dot{-} \mathfrak{M}_T$. ($\mathfrak{M}_T, \mathfrak{M}_T$ are the supplementary components of T and $\mathbf{T}$, respectively.)

2) The operator T, defined on $H \dot{-} \mathfrak{M}_T$, is thereby transformed into its model $\mathbf{T} = UTU^{-1}$, defined on $H \dot{-} \mathfrak{M}_T$.

The theorem follows from the fact that to every quasiunitary operator T one can associate a normalized characteristic matrix-function $w_T(\zeta)$. Using the decomposition of $w_T(\zeta)$ by formula (1), one can construct a triangular model $\mathbf{T}$ by formulas (2) and (3). Since, by Theorem 1, $w_T(\zeta)$ coincides with $w_T(\zeta)$, the operators T and $\mathbf{T}$ are unitary-equivalent up to a supplementary component ([2]).

We note that Theorem 2 does not follow from the results of ([1]), since the Cayley transform of a quasiunitary operator, generally speaking, is not an operator of the class $(i\Omega)$.

4. In the present section the question of completeness of the eigenfunctions and associated functions of the operator T will be solved in the case when $J = I$. In this case the quasiunitary operator is a contraction operator ([5]).

Theorem 3. If T is a quasiunitary contraction of rank r with signature $(r,0)$, then the inequality

$$|\operatorname{Det}\tau| \le \prod_{k=1}^{\infty} |\zeta_k|, \quad (6)$$

holds, where ζ_k are the eigenvalues of the operator T; τ is the matrix determined by the equality

$$\tau = \left\|(Te_k, e'_j)\right\|_{k,j=1}^{r}$$

(e_k, e'_j are orthonormal bases in D_T and D'_T, respectively). The system of eigenfunctions and associated functions of the operator T is complete if and only if equality holds in relation (6).

The proof of this theorem is based on the properties of the triangular model of the operator T.

5. As an application of Theorem 2, let us consider the operator T defined in the Hilbert space H by the formulas

$$Te_k = e_{k-1} \quad (k = \pm 1, \pm 2, ...); \qquad Te_0 = \tau e_{-1}, \quad 0 < \tau < 1,$$

where $\{e_k\}_{k=-\infty}^{\infty}$ is an orthonormal basis of H.

It is not difficult to verify that T is a quasi-unitary operator of rank one, and that the characteristic function of the operator $w_T(\zeta)$ has the form $w_T(\zeta) \equiv \tau$. The function $w_T(\zeta)$ can be represented in the form

$$w_T(\zeta) = \exp\left[-\int_0^{\ln\frac{1}{\tau}} \frac{e^{i\varphi(x)} + \zeta}{e^{i\varphi(x)} - \zeta}\,dx\right], \quad \text{where } \varphi(x) = \frac{2\pi x}{\ln\frac{1}{\tau}}. \tag{7}$$

Therefore the triangular model of the operator T has the form

$$Tf(x) = f(x)e^{i\varphi(x)} - 2\int_x^{\ln\frac{1}{\tau}} f(t)e^{x-t}e^{i\varphi(x)}\,dt, \tag{8}$$

where $\varphi(x)$ is defined by formula (7).

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Note: Figure translations are in progress. See original paper for figures.