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Abstract

Full Text

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ON GROUPS WITH FINITE CLASSES OF CONJUGATE ELEMENTS

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In the present note a new definition of the class of layer-finite groups is given (Theorem 1), and, in addition, certain properties are established of groups with finite classes of conjugate elements, connected with the existence in them of Abelian subgroups with centralizers of finite index.

1. Theorem 1. *The class of locally normal groups with special Sylow subgroups (i.e., with Sylow subgroups satisfying the minimal condition for Abelian subgroups) coincides with the class of layer-finite groups.*

Proof. In the author' s work ([1]) it was established that the second of these classes is contained in the first. To prove that the first class is contained in the second, consider an arbitrary locally normal group G with special Sylow subgroups. Let A be its maximal complete subgroup. Since the normalizer of every element of a locally normal group has finite index in it, and since no complete group has proper subgroups of finite index, the intersection of the subgroup A with the normalizer of an arbitrary element of the group G coincides with A. It follows from this that the subgroup A is contained in the center of the group G.

Since an arbitrary Sylow subgroup P of the group G, being a special group, is a finite extension of its maximal complete subgroup, the intersection $P \cap A$, in view of the maximality of the complete subgroup A, will have finite index in P. Hence it is easily obtained that all Sylow subgroups of the factor group G/A are finite (the subgroup A is invariant in G, since, as was proved, it is contained in the center of G). Indeed, this follows from the fact that if P is a Sylow subgroup of the group G, then PA/A is a finite Sylow subgroup of the group G/A, and from the fact that all Sylow p-subgroups of a locally normal group for fixed p are conjugate to one another, if at least one of them is finite.

From the finiteness and conjugacy of the Sylow p-subgroups of the group G/A and its local normality there follows the finiteness of the set of its Sylow p-subgroups for each fixed prime number p. Hence one readily sees the finiteness of the set of all p-elements of the group G/A (i.e., the finiteness of the set of all its elements of order p^k, where k is an arbitrary natural number).

Let, further, R/A be the subgroup generated by the set of all p-elements of the group G/A. In view of the local normality of the group G/A and the finiteness of this set, it is finite. Since the group R/A contains all p-elements of the group

G/A, all p-elements of the group G will be contained in the subgroup R. If one chooses one element from each coset of the group R modulo the subgroup A, then the group R^*, generated by

chosen elements, will be finite, and the product $\mathfrak{A}\mathfrak{R}^*$ coincides with the group $\mathfrak{R}$. Since the group $\mathfrak{A}$ is, evidently, layer-finite and its elements commute with the elements of the subgroup $\mathfrak{R}^*$, it follows from the decomposition $\mathfrak{R} = \mathfrak{A}\mathfrak{R}^*$ that the set of elements of the group $\mathfrak{R}$, and hence also of the group $\mathfrak{G}$, having one and the same order p^k, is finite for an arbitrary natural number k. In view of the arbitrariness of the prime number p, this means that the group $\mathfrak{G}$ is layer-finite. The theorem is proved.

Corollary 1. *A locally normal group satisfying the minimal condition for abelian subgroups is layer-finite.*

Corollary 2. *If a locally normal group satisfies the minimal condition for abelian subgroups, then it is either finite, or is a finite extension of a direct product of a finite set of quasicyclic groups.*

Proof. Since a locally normal group satisfying the minimal condition for abelian subgroups has special Sylow subgroups, it is layer-finite. But then, in view of Theorem 12 of paper ([1]), the set of prime divisors of the orders of its elements is finite. Hence, by Theorem 1 of the same paper, the assertion to be proved follows.

Corollary 3. *If all abelian subgroups of a locally normal group are finite, then the group itself is finite.*

2. Lemma. *If an abelian subgroup $\mathfrak{A}$ of a group $\mathfrak{G}$ with finite conjugacy classes of elements has in $\mathfrak{G}$ a centralizer of finite index, then its intersection with the center $\mathfrak{Z}$ of the group $\mathfrak{G}$ has finite index in it.*

Proof. Let $z(\mathfrak{A})$ be the centralizer of the subgroup $\mathfrak{A}$ in $\mathfrak{G}$, and let M be the set consisting of representatives of the distinct cosets $Xz(\mathfrak{A})$, one chosen from each coset, and let $z(M)$ be its centralizer in $\mathfrak{G}$. From the finiteness of the set M and the finiteness of all conjugacy classes of elements of the group $\mathfrak{G}$ it follows that $z(M)$ has finite index in $\mathfrak{G}$. Hence it follows that the intersection $\mathfrak{A} \cap z(M)$ has finite index in $\mathfrak{A}$. Since it is evident that $\mathfrak{A} \cap z(M) \subset \mathfrak{Z}$, our assertion is proved.

Corollary. *If all conjugacy classes of elements of the group $\mathfrak{G}$ are finite, then the factor group of the group $\mathfrak{G}$ by its center is periodic.*

If the condition of the lemma is satisfied, in particular, for a maximal abelian subgroup $\mathfrak{A}$ of the group $\mathfrak{G}$, then the latter, by the lemma, will be a finite extension of its center. Observing that in groups which are finite extensions of their center there do not exist abelian subgroups with centralizer of infinite index, we obtain the following assertion.

Theorem 2. *In a group $\mathfrak{G}$, the centralizer of every abelian subgroup has finite index if and only if it is a finite extension of its center.*

Remark. If $\mathfrak{G}$ is a torsion-free group, then, in view of Theorem 5.4 of paper ([2]), it will simply be abelian.

3. Theorem 3. *If in an infinite periodic group $\mathfrak{G}$, having no elements of infinite height, the centralizer of each of its abelian subgroups has finite index, then it is the direct product of an infinite abelian group and some finite group.*

A nonidentity element X of the group $\mathfrak{G}$ is called here an element of **infinite height** in it if, for every natural number n, one can choose elements in the group $\mathfrak{G}$ such that the element X is expressible as a product of n-th powers of these elements.

Proof. Choosing one representative from each coset of the group $\mathfrak{G}$ by its center $\mathfrak{Z}$, and considering the finite subgroup $\mathfrak{R}$ generated by them, we obtain the decomposition $\mathfrak{G} = \mathfrak{Z}\mathfrak{R}$. Since the group $\mathfrak{Z}$ has no elements of infinite height and the intersection $\mathfrak{Z} \cap \mathfrak{R}$ is finite, there exists a decomposition $\mathfrak{Z} = \mathfrak{Z}_1 \times \mathfrak{Z}_2$, in which $\mathfrak{Z}_1$ is a finite group,

containing the intersection $\mathfrak{Z} \cap \mathfrak{R}$. But then the decomposition $\mathfrak{G} = \mathfrak{Z}_2 \times (\mathfrak{Z}_1\mathfrak{R})$ will hold, proving our assertion.

Theorem 4. *If the centralizer of every abelian p-subgroup of a periodic group $\mathfrak{G}$ has finite index in $\mathfrak{G}$, then the factor group of the group $\mathfrak{G}$ by its center is a locally finite thin group.*

Proof. In view of Theorem 2, the center $\mathfrak{Z}(\mathfrak{P})$ of an arbitrary Sylow subgroup $\mathfrak{P}$ of the group $\mathfrak{G}$ has finite index in $\mathfrak{P}$. Since the centralizer $z(\mathfrak{Z}(\mathfrak{P}))$ of the subgroup $\mathfrak{Z}(\mathfrak{P})$ has finite index in $\mathfrak{G}$, and since all conjugacy classes of elements of $\mathfrak{G}$, obviously, are finite, it follows from the lemma that its intersection with the center $\mathfrak{Z}$ of the group $\mathfrak{G}$ is a subgroup of finite index in $\mathfrak{Z}(\mathfrak{P})$. Hence it follows that an arbitrary Sylow subgroup $\mathfrak{P}$ of the group $\mathfrak{G}$ corresponds in the factor group $\mathfrak{G}/\mathfrak{Z}$ to the finite group $\mathfrak{P}\mathfrak{Z}/\mathfrak{Z}$. The latter will, obviously, be a Sylow subgroup of the group $\mathfrak{G}/\mathfrak{Z}$. But then all Sylow subgroups of the group $\mathfrak{G}/\mathfrak{Z}$ are finite. Hence, by Theorem 1, it follows that the group $\mathfrak{G}/\mathfrak{Z}$ is locally finite. Since the locally finite group $\mathfrak{G}/\mathfrak{Z}$ has no infinite Sylow subgroups, it is thin. The theorem is proved.

In the case of an arbitrary group $\mathfrak{G}$ with finite conjugacy classes of elements (even in the case of a periodic group $\mathfrak{G}$) one can assert only that the factor group of the group $\mathfrak{G}$ by its center $\mathfrak{Z}$ (in view of the corollary of the lemma this factor group is periodic) contains no elements of infinite height.

Indeed, the normalizer $z(X)$ of an arbitrary element X of the group $\mathfrak{G}$ under consideration contains its center $\mathfrak{Z}$ and has finite index in it. But then the group $z(X)/\mathfrak{Z}$ will also have finite index in $\mathfrak{G}/\mathfrak{Z}$. If $\mathfrak{N}(X)/\mathfrak{Z}$ is the normal divisor contained in it of finite index in the group $\mathfrak{G}/\mathfrak{Z}$ (its existence follows from Poincaré' s theorem on subgroups of finite index), then the factor group $(\mathfrak{G}/\mathfrak{Z})/(\mathfrak{N}(X)/\mathfrak{Z})$, in view of its finiteness, will not have elements of infinite height, and therefore all elements of infinite height from the group $\mathfrak{G}/\mathfrak{Z}$ are contained in the subgroup

$\mathfrak{R}(X)/\mathfrak{Z}$ and, consequently, in the subgroup $z(X)/\mathfrak{Z}$. In view of the arbitrariness of the element X of $\mathfrak{G}$ and the relation $\mathfrak{Z} = \bigcap_{X\subset\mathfrak{G}} z(X)$, where $\cap$ is the sign of intersection, it follows that each element of infinite height from $\mathfrak{G}/\mathfrak{Z}$ is contained in the intersection

$$\bigcap_{X\subset\mathfrak{G}} z(X)/\mathfrak{Z} = \mathfrak{Z}/\mathfrak{Z},$$

which is impossible, since each element of infinite height is distinct from the identity by its very definition.

From the proposition proved it follows, in particular, that:

1) No factor of the upper central series of a zA-group with finite conjugacy classes of elements, except perhaps the first, contains elements of infinite height.

2) If some factor of the upper central series of a zA-group with finite conjugacy classes of elements, following its first factor, contains only a finite number of elements of prime order, then it is finite.

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2. B. H. Neumann, *Proc. London Math. Soc.* (3), **1**, 178 (1951).

Note: Figure translations are in progress. See original paper for figures.