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S. M. NIKOLSKII

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Abstract

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S. M. NIKOLSKII

ON A VARIATIONAL PROBLEM OF HILBERT

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D. Hilbert, in one of his papers (1), considered the following problem.

In the complex plane R (or on a Riemann surface with a finite number of branch points and a finite number of sheets) a closed curve C is given, and a class $\mathfrak{M}$ of functions f possessing the following properties:

1. The function $f \in \mathfrak{M}$ is defined, continuous, and piecewise analytic on $R - C$.
2. The quantity

$$D[f] = \iint_R \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right] dx\, dy < \infty$$

is finite.

3. On passing from one side of the curve C to the other, the function f undergoes a jump equal to one.
4. The mean value of f over some segment γ, not intersecting C, is equal to zero.

Hilbert proves, restricting himself to considering as the curve C a closed polygon with sides parallel to the axes x and y (not passing, in the case where R is a Riemann surface, through the branch points of R), that in the class $\mathfrak{M}$ there exists a function u for which the minimum is attained

$$\min_{f \in \mathfrak{M}} D[f] = D[u]. \tag{1}$$

Here u is a harmonic function on $R - C$, regular at infinity.

The results formulated below show that similar facts also hold in n-dimensional space (we do not have in mind, however, any generalizations of a Riemann surface). In addition, we investigate the extremal function u and give for it another equivalent definition in terms not connected with the variational problem.

Let g be a domain of the real space R_n, bounded by a twice continuously differentiable surface Λ. Define on $R_n - \Lambda$ a function f having generalized (in the sense of Sobolev) first-order derivatives satisfying the condition

$$D[f] = \int_g \sum_1^n \left(\frac{\partial f}{\partial x_i} \right)^2 dg < \infty. \tag{2}$$

For such a function, for almost all points $Q \in \Lambda$ there exist the limits

$$\lim_{P \to +Q} f(P) = f_+(Q), \qquad \lim_{P \to -Q} f(P) = f_-(Q)$$

respectively from the side of the inner (positive) and outer (negative) normals to Λ.

Denote by H the set of functions $\varphi = f_+$, where f is an arbitrarily prescribed function on $R_n - \Lambda$ satisfying condition (2). Also denote by Λ_δ and $\Lambda_{-\delta}$ the surfaces at distance $\delta > 0$ from Λ along the normals to Λ, respectively on the positive and negative sides. For sufficiently small δ, these surfaces have a meaning under the conditions imposed on Λ.

The layer $V_\delta \in R_n$, bounded by the surfaces Λ_δ and $\Lambda_{-\delta}$, consists of points P whose position is completely determined by a pair (Q, h), where $Q \in \Lambda$ is the point on the normal to Λ on which P lies, and h is a number equal to the distance from P to Q, taken with the corresponding sign.

It can be proved that a function f satisfying condition (2) can be modified on a set of n-dimensional measure zero, and after this

$$\int_\Lambda [f(Q, \delta) - f_+(Q)]^2 \, d\Lambda = o(\delta) \qquad (\delta \to 0,\ \delta > 0).$$

Here $f_+ \in L_2(\Lambda)$, i.e. it has an integrable square on Λ.

Let $a(Q) \neq 0$ and $b(Q)$ be continuously differentiable functions prescribed on Λ, and let $\varkappa(Q) \in H$, and let γ be an $(n-1)$-dimensional parallelepiped lying in R_n but outside Λ.

By definition, a function f belongs to the class $\mathfrak{M}$ if the following properties hold for it:

1) The function f is defined on $R_n - \Lambda$, has generalized derivatives there, and inequality (2) is satisfied.
2) On Λ one has

$$af_+ - bf_- = \varkappa.$$

3)

$$\int_{\gamma} f\,d\gamma = 0.$$

If $\varkappa \equiv 0$ on Λ, then the class $\mathfrak{M}$ will be denoted by $\mathfrak{M}_0$.

Theorem 1. *Among the functions f of the class $\mathfrak{M}$ there exists, moreover a unique, function u for which the minimum of the variational problem (1) is attained. The function u is harmonic on $R_n - \Lambda$.*

The following assertions are equivalent:

a) the function u satisfies the conditions of Theorem 1;

b) the function $u \in \mathfrak{M}$ and for it one has

$$D[u,f] = \int_R \sum_1^n \frac{\partial u}{\partial x_i}\frac{\partial f}{\partial x_i}\,dR = 0$$

for all $f \in \mathfrak{M}_0$;

c) the function $u \in \mathfrak{M}_0$ is harmonic on $R_n - \Lambda$ and for all $\varphi \in H$

$$\lim_{\delta\to 0}\int_{\Lambda}\left\{ b\,\frac{\partial u}{\partial n}(Q,\delta) - a\,\frac{\partial u}{\partial n}(Q,-\delta)\right\}\varphi(Q)\,d\Lambda = 0,$$

$$\int_{\sigma_\rho}\left(\frac{\partial u}{\partial \rho}\right)^2 d\sigma_\rho = O(\rho^{1-n}) \qquad (\rho\to\infty),$$

where $\sigma_\rho \subset R$ is the surface of radius ρ with center at the origin of coordinates;

d) for $n = 2$, the function u, harmonic on $R_n - \Lambda$, belongs to $\mathfrak{M}$, and for any conjugate function v to it, for almost all s (s is the length of the arc Λ) there exist limiting values v_+ and v_- on Λ, satisf–

satisfying the equalities

$$\int_0^l \mu(s)\,ds = 0, \qquad \nu(s) + \int_0^s \mu(t)\,dt + \lambda_0 = 0,$$

where

$$\mu(s) = v\_a' - v_+b', \qquad \nu(s) = v\_a - v_+b,$$

and λ_0 is a constant depending on v.

In the two-dimensional case, the variational problem considered above is equivalent to a special (particular) Hilbert problem in the theory of functions of a complex variable, well studied by means of the apparatus of singular integral equations [2].

Let us note several lemmas on which the proof of the stated assertions was based.

Lemma 1. For every ball $\omega \subset R$ there exists a constant C_ω such that, for all $f \in \mathfrak{M}_0$,

$$\|f\|_{L_2(\omega)} = \left(\int_\omega |f|^2 \, d\omega \right)^{1/2} < C_\omega \sqrt{D[f]}.$$

Lemma 2. For a function f with finite $D[f]$ the estimates

$$\int_{\sigma_\rho} f^2 \, d\sigma_\rho < \begin{cases} C\sqrt{\rho \ln \rho}, & \text{for } n = 2, \\ C\sqrt{\rho^{n-1}}, & \text{for } n > 2, \end{cases} \qquad (\rho \geq 1)$$

hold, where σ_ρ is the surface of the ball of radius ρ with center at the origin, and C does not depend on ρ.

Lemma 3. Let $u(P)$ be a continuous function defined in a domain g with boundary Λ, satisfying the conditions stated above. Moreover,

$$\int_g u^2 \, dg = M^2$$

and there exists a constant K such that, if S is a spherical surface with center at an arbitrary point $P \in g$, entirely contained in g, then the inequality

$$|u(P)| \leq \frac{K}{|S|} \int_{|S|} |u| \, dS$$

holds, where $|S|$ is the area of S. Then there exists a constant C_g such that, for sufficiently small $\delta > 0$,

$$\int_\Lambda u^2(Q; h) \, d\Lambda < C^2_{gKM} h^{-1} \qquad (0 < h < \delta).$$

In conclusion I wish to express my gratitude to A. V. Bitsadze, whose general considerations on Hilbert' s problem, expressed by him in our conversation, were very valuable to me.

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CITED LITERATURE

[1] D. Hilbert, *Math. Ann.*, **59**, 161 (1904); D. Hilbert, *Gesammelte Abhandlungen*, 3, 1935, pp. 15–27.
[2] N. I. Muskhelishvili, *Singular Integral Equations*, 1946.

Note: Figure translations are in progress. See original paper for figures.

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