MATHEMATICS

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1957

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Abstract

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MATHEMATICS

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ON THE A. CARTAN ALGEBRA OF A POLYNOMIAL IDEAL

(Presented by Academician P. S. Aleksandrov, 19 XI 1956)

1. Definitions and results. In the ring of polynomials $R = R(x_1, \dots, \dots, x_m)$ over a certain field, consider an ideal $I = I(F_1, \dots, F_n)$ with generators $F_1, \dots, F_n$. Let $\Lambda = \Lambda(z_1, \dots, z_n)$ be the exterior algebra with generators $z_1, \dots, z_n$ (i.e., the algebra with defining relations $z_i z_j = -z_j z_i$). We grade* the algebra $C = R \otimes \Lambda$ by setting $\deg x_i = l_i$, $\deg z_j = k_j$ (l_i, k_j are positive integers), and require that $F_1, \dots, F_n$ be homogeneous (in the sense of this grading) of degrees, respectively, $k_1, \dots, k_n$. Introduce in C a differential Δ, acting in the following way:

$$\Delta(r \otimes z_{i_1} \dots z_{i_k}) = \sum_{\alpha=1}^{k} (-1)^\alpha r F_{i_\alpha} \otimes z_{i_1} \dots z_{i_{\alpha-1}} z_{i_{\alpha+1}} \dots z_{i_k},$$

$$\Delta(r \otimes 1) = 0, \quad \text{where } r \in R(x_1, \dots, x_m).$$

Under these conditions we shall call the differential graded algebra C the **Cartan algebra** ([1]) of the given ideal I and denote it by $C(F_1, \dots, F_n)$. The differential Δ, acting in C, generates the homology algebra**

$$H = \operatorname{Ker} \Delta / \operatorname{Im} \Delta = H(C) = H(F_1, \dots, F_n).$$

If, in general, $A = \sum_q A^{(q)}$ is a graded algebra over the given field, then we denote by $A(t)$ the polynomial (or series) $\sum_q a_q t^q$, where a_q is the dimension of the vector subspace $A^{(q)}$ of the algebra A.

There is a direct decomposition $C = \sum_q C^{(q)}$, where $C^{(q)} = R \otimes \Lambda^q$ is the subspace C of exterior degree q (([2]), § 11). To it corresponds the decomposition $H = \sum_q H^{(q)}$, where $H^{(q)} = H(C^{(q)})$. Hence it is obvious that

$$H(t) = H^{(0)}(t) + H^{(1)}(t) + \dots + H^{(n)}(t).$$

The coefficients of $H^{(0)}(t)$ express, for each degree p, the dimension of the quotient space R/I. Their values constitute the so-called Hilbert function (of p) ([4]).

* $\deg c$ denotes the degree of the element c.
** If φ is a mapping of M into M', then $\operatorname{Ker}\varphi$ denotes the kernel of φ; $\operatorname{Im}\varphi$, the image of φ; $\operatorname{Coker}\varphi = M'/\operatorname{Im}\varphi$.

A relation among $F_1, \dots, F_n$ is an equality of the form $P_1F_1 + P_2F_2 + \dots + P_{nF}n = 0$, where $P_1, \dots, P_n$ are polynomials. Relations of the form $(PF_i)F_j + (-PF_j)F_i = 0$, and those generated by them, will be regarded as trivial.

It is easy to see that for a trivial relation $P_i = \sum_{j=1}^{n} P_{ij}F_j$, where the P_{ij} are polynomials such that $P_{ij} = -P_{ji}$. The coefficients of the polynomial $H^{(1)}(t)$ describe the dimensions of the quotient module of all relations among $F_1, \dots, F_n$ by those of them which are trivial. We shall call the polynomials $F_1, \dots, F_n$ **completely independent** if there are no relations among them other than the trivial ones.*

The principal results of the paper are formulas (4) and (5). Relying on these formulas, in a number of cases one can express $H(t)$ in terms of $H^{(0)}(t)$. We shall start from completely independent $F_1, \dots, F_n$, whose number is equal to the number of unknowns: $m = n$. According to property 1) of § 2, $H(F_1, \dots, F_n) = H^{(0)}$. Let $\widetilde{F}_k$ be the polynomials obtained from F_k for $x_n = 0$. Then for the algebra $H(\widetilde{F}_1, \dots, \widetilde{F}_n)$ we have: $H(t) = H^{(0)}(t) + H^{(1)}(t)$, where

$$H^{(0)}(t) - H^{(1)}(t) = \frac{(1-t^{k_1})\dots(1-t^{k_n})}{(1-t^{l_1})\dots(1-t^{l_{n-1}})}. \tag{A}$$

Next, let $\widetilde{\widetilde{F}}_k$ be obtained from F_k for $x_{n-1} = x_n = 0$. Then for the algebra $H(\widetilde{\widetilde{F}}_1, \dots, \widetilde{\widetilde{F}}_n)$ we shall have: $H(t) = H^{(0)}(t) + H^{(1)}(t) + H^{(2)}(t)$, where

$$H^{(0)}(t) - H^{(1)}(t) + H^{(2)}(t) = \frac{(1-t^{k_1})\dots(1-t^{k_n})}{(1-t^{l_1})\dots(1-t^{l_{n-2}})}, \quad H^{(2)}(t) = t^M H^{(0)}\left(\frac{1}{t}\right), \tag{B}$$

where $M = k_1 + k_2 + \dots + k_n - l_1 - l_2 - \dots - l_{n-2}$.

As was shown in Cartan' s paper ([1]) (see also ([2]), § 25), the cohomology algebra (over the field of rational numbers) of any homogeneous space $\mathfrak{M} = \mathfrak{G}/\mathfrak{G}'$ of conjugacy classes of a compact Lie group $\mathfrak{G}$ (of rank n) with respect to its closed subgroup $\mathfrak{G}'$ (of rank $m = n - s$) coincides with the algebra $H(F_1, \dots, F_n)$ under a suitable choice of the polynomials $F_1, \dots, F_n$. In topological applications the degrees l_i and $k_j = \deg F_j$ are even, and (in distinction to the grading introduced by us) $\deg z_j = k_j - 1$. Therefore, in our notation the Poincaré polynomial of a homogeneous space takes the form:

$$\Pi(t) = \sum_q \frac{1}{t^q} H^{(q)}(t).$$

In cases where the rank of the subgroup $\mathfrak{G}'$ is smaller by 1 or by 2 than the rank of the group $\mathfrak{G}$, one can show that formulas (A) and (B) are applicable and, consequently, the polynomial $\Pi(t)$ is completely determined by its part $H^0(t)$ (i.e. by the characteristic subalgebra of the cohomology algebra of the homogeneous space ([1,2])).

In what follows we shall consider the Cartan algebra and its cohomology algebra as graded vector spaces, disregarding their multiplicative structure. By homomorphisms we shall mean linear mappings (but not homomorphisms of algebras).

2. Complete independence of polynomials. It is easy to see that complete independence (see above) is invariant under a linear change of the unknowns and, for a given ideal, does not depend on the choice of generators. We note several properties connected with this notion.

* This definition is preferable to that given in ([5]). It is not difficult to show that complete independence is equivalent to the absence of relations in Borel' s sense ([2]) (but Borel' s very notion of relation differs from ours).

1) If $F_1, \dots, F_n$ are completely independent, then

$$H^{(i)}(F_1, \dots, F_n) = 0 \quad \text{for } i \geqslant 1, \qquad H^{(0)}(t) = \frac{(1-t^{k_1}) \cdots (1-t^{k_n})}{(1-t^{l_1}) \cdots (1-t^{l_n})}. \tag{1}$$

(see ([3]), § 58; ([4]), §§ 142 and 152).

2) For $n = m$ the series (1) (for completely independent $F_1, \dots, F_n$) is finite, i.e. is a polynomial in t.

3) The number of completely independent polynomials cannot exceed the number of unknowns: $n \leqslant m$.

4) Let $G_1, \dots, G_n$ be arbitrary polynomials in $x_1, \dots, x_m$. Then one can find completely independent polynomials $F_1, \dots, F_m$ in $x_1, x_2, \dots, x_m, x_{m+1}, \dots, x_{m+p}$, from which $G_1, \dots, G_n$ are obtained by setting $x_{m+1} = \dots = x_{m+p} = 0$. In general one cannot require that the degrees of $x_{m+1}, \dots, x_{m+p}$ be prescribed in advance (if we require the homogeneity of $F_1, \dots, F_n$), but they may be taken, for example, equal to 1.

3. The Koszul module. Let several linear operators $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_s$ act in a vector space R, commuting with one another: $\varepsilon_i \varepsilon_j = \varepsilon_j \varepsilon_i$. Let, further, $\Lambda = \Lambda(z_1, \dots, z_s)$ be the exterior algebra with generators $z_1, \dots, z_s$. In the space $R \otimes \Lambda$ introduce a differential Δ, putting:

$$\Delta(r \otimes z_{i_1} \cdots z_{i_t}) = \sum_{\alpha=1}^{t} \varepsilon_{i_\alpha} r \otimes z_{i_1} \cdots z_{i_{\alpha-1}} z_{i_{\alpha+1}} \cdots z_{i_t},$$

$$\Delta(r \otimes 1) = 0, \qquad r \in R.$$

The differential space so obtained $C = C(R, \varepsilon_1, \dots, \varepsilon_s)$ will be called the K-module of the space R ([6]); its homology space will be denoted by $H(C) = H(R, \varepsilon_1, \dots, \varepsilon_s)$. The previously introduced space of the Cartan algebra is a special case, when R is the space of polynomials and ε_i is the operator of multiplication by F_i.

Homomorphism of embedding. Denote $C_{(p)} = C(R, \varepsilon_1, \dots, \varepsilon_p)$, $H_{(p)} = H(C_{(p)})$ $(p \leqslant s)$. The operator ε_i $(i > p)$ acts in $C_{(p)}$ by the formula $\varepsilon_i(r \otimes z) = \varepsilon_i r \otimes z$. Since $\varepsilon_i \Delta = \Delta \varepsilon_i$, ε_i also acts in $H_{(p)}$. Further denote by $H^{(q)}_{(p)} = H(C^{(q)}_{(p)})$, where $C^{(q)}_{(p)}$ is the subspace of $C_{(p)}$ of exterior degree q.

Consider the embedding $\xi : H_{(p)} \to H_{(p+1)}$. It is easy to see that $H_{(p+1)} = (H_{(p)}, \varepsilon_{p+1})$. Hence the isomorphisms are obtained:

$$\operatorname{Ker} \xi(H^{(q)}_{(p)}) \simeq \operatorname{Im} \varepsilon_{p+1}(H^{(q)}_{(p)}); \qquad \operatorname{Coker} \xi(H^{(q)}_{(p+1)}) \simeq \operatorname{Ker} \varepsilon_{p+1}(H^{(q-1)}_{(p)}) \tag{2}$$

(in parentheses are indicated the spaces with respect to which the corresponding operator is considered).

Duality. Consider the space R^*, dual to R, and the operators ε_i^*, adjoint to ε_i, acting in R^*. Let $\Lambda(\zeta_1, \dots, \zeta_s)$ be the exterior algebra with generators $\zeta_1, \dots, \zeta_s$. Introduce a scalar product between $R \otimes \Lambda(z_1, \dots, z_s)$ and $R^* \otimes \Lambda(\zeta_1, \dots, \zeta_s)$, putting: $(r \otimes z_{i_1} \cdots z_{i_k}, r^* \otimes \zeta_{j_1} \cdots \zeta_{j_l}) = \pm(r, r^*)$, if $(i_1 \dots i_k, j_1 \dots j_l)$ is a permutation of $1, 2, \dots, n$; the sign is chosen depending on its parity. In all other cases we set the scalar product equal to zero.

Let Δ^* be the operator adjoint to Δ with respect to this scalar product. It is proved that Δ^* acts in $R^* \otimes \Lambda(\zeta_1, \dots, \zeta_s)$ with respect to the operators $\varepsilon_1^*, \dots, \varepsilon_s^*$ by the same formulas (up to sign) as Δ. Hence one obtains the isomorphism

$$H^{(q)}(R, \varepsilon_1, \dots, \varepsilon_s) \simeq H^{(s-q)}(R^*, \varepsilon_1^*, \dots, \varepsilon_s^*). \tag{3}$$

4. **The theorem on the alternating sum.** For an arbitrary Cartan algebra $C(F_1, \dots, F_n)$ the formula

$$\sum_q (-1)^q H^{(q)}(t) = \frac{(1 - t^{k_1})(1 - t^{k_2}) \cdots (1 - t^{k_n})}{(1 - t^{l_1})(1 - t^{l_2}) \cdots (1 - t^{l_m})}. \tag{4}$$

holds. If the generators of the ideal are completely independent, then (4) reduces to (1). In the general case formula (4) is proved by induction on n, using the exact sequence of the embedding $0 \to \operatorname{Ker} \xi \to H_{(n-1)} \to H_{(n)} \to \operatorname{Coker} \xi \to 0$ and the isomorphisms (2).

5. **Reduction to the quotient space.** Let $F_1, \dots, F_n(x_1, \dots, x_m)$ be completely independent; $\widetilde{F}_1, \dots, \widetilde{F}_n$ are obtained from them by setting $x_1 = x_2 = \dots = x_s = 0$. Denote

$$L = \frac{R(x_1, \dots, x_m)}{I(F_1, \dots, F_n)}.$$

We shall regard the unknowns $x_1, \dots, x_s$ as operators acting in L.

Theorem. There is an isomorphism

$$H^{(q)}(\widetilde{F}_1, \dots, \widetilde{F}_n) \cong H^{(q)}(L, x_1, \dots, x_s),$$

where $H^{(q)}(\widetilde{F}_1, \dots, \widetilde{F}_n)$ is the homology space of the Cartan algebra of the ideal $(\widetilde{F}_1, \dots, \widetilde{F}_n)$ in the ring $R(x_{s+1}, \dots, x_m)$; $H^{(q)}(L, x_1, \dots, x_s)$ is the homology space of the K-module of the space L relative to the operators $x_1, \dots, x_s$.

Corollary. $H^{(q)}(\widetilde{F}_1, \dots, \widetilde{F}_n) = 0$ for $q > s$.

6. **Duality in the homologies of the Cartan algebra.** Consider an ideal I with completely independent generators whose number is equal to the number of unknowns: $n = m$. In the space $R^*(x_1, \dots, x_n)$, dual to $R(x_1, \dots, x_n)$, consider the annihilator L^* of the ideal $I(F_1, \dots, F_n)$. It is invariant with respect to the operators x_i^*, conjugate to the multiplication operators x_i. Clearly, $L = R/I$ and L^* may be regarded as dual spaces in which x_i and x_i^* will be conjugate operators.

It is easy to see that in the space L^* there is a unique (up to a factor) element Ω of degree $N = k_1 + k_2 + \dots + k_n - l_1 - \dots - l_n$. Let P be an element of L of degree p; we assign to it an element of L^* of degree $N - p$ in the following way: if $\widetilde{P}$ is a representative of P, then put $\eta(P) = \widetilde{P}^*\Omega$, where $\widetilde{P}^*$ is the operator conjugate to the operator of multiplication by $\widetilde{P}$. It can be proved that η is an isomorphism of vector spaces, and moreover $\eta x_i = x_i^* \eta$ for the multiplication operator x_i. Consequently, η realizes an isomorphism of the homology spaces

$$H^{(q)}(L, x_1, \dots, x_s) \quad \text{and} \quad H^{(q)}(L^*, x_1^*, \dots, x_s^*).$$

In view of the duality of item 3 we obtain the isomorphism

$$H^{(q)}(L, x_1, \dots, x_s) \cong H^{(s-q)}(L, x_1, \dots, x_s).$$

Hence the following result is obtained:

Theorem. Suppose that the generators of the ideal $I(\widetilde{F}_1, \dots, \widetilde{F}_n)$ are obtained from n completely independent polynomials in n unknowns if one sets $x_n = x_{n-1} = \dots = x_{n-s+1} = 0$. Then

$$H^{(s-q)}(t) = t^M H^{(q)}\left(\frac{1}{t}\right), \tag{5}$$

where

$$M = k_1 + k_2 + \cdots + k_n - l_1 - l_2 - \cdots - l_m$$

(in the notation of item 1).

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Received
13 XI 1956

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.