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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

HYDROMECHANICS

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ON UNSTEADY MOTIONS OF AN INCOMPRESSIBLE FLUID IN NARROW REGIONS

(Presented by Academician M. A. Lavrent’ ev on 30 III 1957)

Consider a plane-parallel unsteady potential motion of an incompressible fluid with a free surface in a channel with a flat bottom. Introduce the following notation: $z = x + iy$—the physical plane; φ—the velocity potential; p_0—the pressure on the free surface; g—the acceleration of gravity; n—the exterior normal; S—arc length; V_n—the projection of the velocity on the normal; ρ_0—the density (see Fig. 1). The problem reduces to finding a harmonic function φ in the region bounded by the curve $y(x, t)$ and by the bottom, under the following boundary conditions:

$$\left.\begin{aligned} \frac{\partial\varphi}{\partial t} &= -\frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right] - gy, \\ \frac{\partial y}{\partial t} &= \frac{1}{\cos\theta}\,\frac{\partial\varphi}{\partial n} \end{aligned}\right\} \quad y = y(x,t) \qquad \begin{aligned} &\text{condition of constant pressure, (1)} \\ &\text{condition of no flow-through;} \qquad (2) \end{aligned}$$

$$\frac{\partial\varphi}{\partial n} = 0 \qquad \text{for } y = 0 \text{ the condition on the channel bottom.} \tag{3}$$

To solve this problem we shall apply the method used by M. A. Lavrent’ ev for solving problems connected with steady flows of an incompressible fluid in narrow regions bounded by a free surface.

Fig. 1

Suppose that the flow region satisfies the conditions

$$kh < y(x,t) < Ch; \qquad \left|\frac{\partial y}{\partial x}\right| < k_1 h^{1/2}; \qquad \left|\frac{\partial^2 y}{\partial x^2}\right| < k_2 h; \qquad \left|\frac{\partial^3 y}{\partial x^3}\right| < k_3 h^{1/2}$$

for one and the same value of the time t, where $h = h(t)$ is a small quantity; k, k_1, k_2, k_3 and C are constants that may depend on the time t.

Then, at each fixed instant of time, we are in the conditions of applicability of M. A. Lavrent’ ev’ s formula, which gives an approximate conformal mapping of the flow region onto an infinite strip of width h (the ζ-plane). In this case the modulus of the boundary derivative is equal to

$$\left|\frac{d\zeta}{dz}\right|_{\substack{y=y(x,t)\\ t=\text{const}}} = \frac{h}{y(x,t)}\left\{1+\frac{1}{3}y\,\frac{\partial^2 y}{\partial x^2}\right\}+R; \qquad |R| < Ah^{1/2},$$

where A depends only on the constants k, k_1, k_2, k_3 and C.

In the plane $\zeta = \xi + i\eta$ the flow region occupies the strip $0 < \eta < h$. Consider the strip $-h < \eta < h$. Let, on its boundaries,

$$\varphi(\xi + ih) = \varphi(\xi - ih);$$

then Schwarz’ s formula for the Dirichlet problem for the strip $-h < \eta < h$ has the form

$$\varphi = \operatorname{Re}\frac{1}{2h}\int_{-\infty}^{+\infty}\frac{\Phi(\zeta)}{\operatorname{ch}\frac{\pi}{2h}(\xi-\zeta)}\,d\xi, \qquad \text{where } \Phi(\xi) = \varphi(\xi \pm ih);$$

the condition on the bottom is automatically satisfied: $\partial\varphi/\partial n = \partial\varphi/\partial\eta = 0$.

Since

$$V_n = \frac{\partial\varphi}{\partial n} = \frac{\partial\varphi}{\partial\eta}\left|\frac{\partial\zeta}{\partial z}\right|$$

on the surface, in order to find V_n it is necessary to compute $\partial\varphi/\partial\xi$. For this purpose we write

$$\varphi(\xi \pm ih) = \varphi_0 + (\xi-\xi_0)\left(\frac{\partial\varphi}{\partial\xi}\right)_{\xi=\xi_0} + \frac{1}{2}(\xi-\xi_0)^2\left(\frac{\partial^2\varphi}{\partial\xi^2}\right)_{\xi=\xi_0} + \cdots \quad \text{for } t = \text{const}.$$

For calculating the derivative, φ_0 is immaterial, since it gives a constant value of the potential throughout the whole region. Substitute the expansion of φ into Schwarz’ s integral. In doing so we shall not write φ_0 and the terms with odd powers $(\xi-\xi_0)^{2n+1}$, since the latter give odd functions under the integral, and therefore the integral from $-\infty$ to $+\infty$ of these terms is equal to zero:

$$\varphi = \operatorname{Re} \frac{1}{2h} \int_{-\infty}^{+\infty} \frac{\sum\limits_{n=1}^{\infty} (\xi - \xi_0)^{2n} \dfrac{1}{(2n)!} \dfrac{\partial^{2n}\varphi}{\partial \xi^{2n}}}{\operatorname{ch} \frac{\pi}{2h}(\xi - \xi_0 - i\eta)} \, d\xi =$$

$$= \sum_{n=1}^{\infty} \left(\frac{2}{\pi}\right)^{2n} \frac{h^{2n}}{\pi(2n)!} \frac{\partial^{2n}\varphi}{\partial \xi^{2n}} \operatorname{Re} \int_{-\infty}^{+\infty} \frac{\left[\frac{\pi}{2h}(\xi - \xi_0)\right]^{2n} \, d\frac{\pi}{2h}(\xi - \xi_0)}{\operatorname{ch}\left[\frac{\pi}{2h}(\xi - \xi_0) - i\frac{\pi}{2h}\eta\right]};$$

$$\frac{\partial}{\partial \eta} \operatorname{Re} \int_{-\infty}^{+\infty} \frac{\left[\frac{\pi}{2h}(\xi - \xi_0)\right]^{2n} \, d\frac{\pi}{2h}(\xi - \xi_0)}{\operatorname{ch}\left[\frac{\pi}{2h}(\xi - \xi_0) - i\frac{\xi}{2h}\eta\right]} = -\frac{\pi^{2n+1}}{h}(2^{2n} - 1)|B_{2n}|,$$

where B_{2n} are the Bernoulli numbers.

Then

$$\left.\frac{\partial \varphi}{\partial \eta}\right|_{\xi_0} = -\sum_{n=1}^{\infty} h^{2n-1} \frac{2^{2n}(2^{2n} - 1)}{(2n)!} |B_{2n}| \left.\frac{\partial^{2n}\varphi}{\partial \xi^{2n}}\right|_{\xi_0} = -\left. h \frac{\partial^2 \varphi}{\partial \xi^2}\right|_{\xi_0} + O(h^4).$$

Hence

$$\frac{\partial \varphi}{\partial n} = \frac{\partial \varphi}{\partial \eta} \left|\frac{\partial \zeta}{\partial z}\right| = -\frac{h^2}{y} \left(1 + \frac{1}{3} y \frac{\partial^2 y}{\partial x^2}\right) \frac{\partial^2 \varphi}{\partial \xi^2} + O(h^{1/2}) =$$

$$= -\frac{h^2}{y} \left(1 + \frac{1}{3} \frac{h^2}{y} \frac{\partial^2 y}{\partial \xi^2}\right) \frac{\partial^2 \varphi}{\partial \xi^2} + O(h^{1/2}).$$

Moreover,

$$\left(\frac{\partial \varphi}{\partial x}\right)^2 = \left(\frac{\partial \varphi}{\partial \xi} \frac{\partial \xi}{\partial S} \frac{\partial S}{\partial x}\right)^2 = \left[\frac{\partial \varphi}{\partial \xi} \left|\frac{\partial \zeta}{\partial z}\right| \cos\theta\right]^2 = \frac{h^2}{y^2} \left(1 + \frac{2}{3} \frac{h^2}{y} \frac{\partial^2 y}{\partial \xi^2}\right) \left(\frac{\partial \varphi}{\partial \xi}\right)^2 + O(h^{1/2}).$$

Since $\left(\frac{\partial \varphi}{\partial y}\right) \sim \left(\frac{\partial \varphi}{\partial \eta}\right) \sim h^4$, we shall neglect this term in condition (1). Substituting the derivatives of φ into conditions (1) and (2), we obtain equations for the unknown functions $y(\xi, t)$ and $\varphi(\xi, t)$:

$$\frac{\partial \varphi}{\partial t} = -\frac{1}{2} \frac{h^2}{y^2} \left(1 + \frac{2}{3} \frac{h^2}{y} \frac{\partial^2 y}{\partial \xi^2}\right) \left(\frac{\partial \varphi}{\partial \xi}\right)^2 - gy + O(h^{5/2}),$$

$$\frac{\partial y}{\partial t} = -\frac{h^2}{y} \left(1 + \frac{1}{3} \frac{h^2}{y} \frac{\partial^2 y}{\partial \xi^2}\right) \frac{\partial^2 \varphi}{\partial \xi^2} + O(h^{7/2}). \tag{4}$$

We have written the equations in the plane (ξ, t); now let us write them in the plane (x, t). To this end, note that

$$\frac{\partial}{\partial \xi}=\frac{\partial x}{\partial S}\frac{\partial S}{\partial \xi}\frac{\partial}{\partial x}=\cos\theta\left|\frac{\partial z}{\partial \xi}\right|\frac{\partial}{\partial x}=\left[1-\frac{1}{2}\left(\frac{dy}{dx}\right)^2\right]\frac{y}{h}\left(1-\frac{1}{3}y\frac{d^2y}{dx^2}\right)\frac{\partial}{\partial x}+O\left(h^{7/2}\right).$$

We now replace in our equations all terms $\partial^k/\partial\xi^k$ by $\partial^k/\partial x^k$, discarding small quantities of higher orders. After the replacement we obtain, in the same approximation,

$$\begin{gathered}\frac{\partial\varphi}{\partial t}=-\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2-gy+O(h^4),\\ \frac{\partial y}{\partial t}=-y\left(1-\frac{1}{3}y\frac{\partial^2 y}{\partial x^2}\right)\frac{\partial^2\varphi}{\partial x^2}-\frac{\partial y}{\partial x}\frac{\partial\varphi}{\partial x}+\frac{1}{3}y^2\frac{\partial^3 y}{\partial x^3}\frac{\partial\varphi}{\partial x}+O\left(h^{7/2}\right).\end{gathered}\tag{5}$$

In practice there may occur cases in which the flow region ω—the physical plane —is not a narrow strip, but a conformal mapping of the narrow strip of the z-plane onto the flow region of the plane ω is known: $\omega = f(z)$. Then, if S_ω is the arc length in the physical plane, then

$$\frac{\partial\varphi}{\partial S_\omega}=\frac{\partial\varphi}{\partial S}\frac{\partial S}{\partial S_\omega}=\left|\frac{1}{f'(z)}\right|\frac{\partial\varphi}{\partial x}+O(h^3).$$

If the force of gravity is not taken into account (it can be taken into account), the equations take the form

$$\begin{gathered}\frac{\partial\varphi}{\partial t}=-\frac{1}{2}\left|\frac{1}{f'(z)}\right|^2\left(\frac{\partial\varphi}{\partial x}\right)^2,\\ \frac{\partial y}{\partial t}=-\left|\frac{1}{f'(z)}\right|^2\left[y\left(1-\frac{1}{3}y\frac{\partial^2 y}{\partial x^2}\right)\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial y}{\partial x}\frac{\partial\varphi}{\partial x}-\frac{1}{3}y^2\frac{\partial^3 y}{\partial x^3}\frac{\partial\varphi}{\partial x}\right].\end{gathered}\tag{6}$$

Integration of the nonlinear systems of partial differential equations (4), (6) apparently presents considerable difficulties. Therefore, of primary interest are the cases when this system of partial differential equations reduces to ordinary equations—these are the investigated cases of steady flows in narrow regions and the cases of self-similar formulations.

Let us illustrate what has been said by the example of a self-similar problem. Consider, in the physical plane ω, a thin liquid wedge impinging with velocity V on an absolutely rigid wedge, whose plane of symmetry in the plane ω is represented by the negative part of the real axis. The plane of symmetry of the liquid wedge then corresponds to the positive part of the real axis. The problem is self-similar, since it is characterized only by two dimensional quantities V and ρ.

Taking the plane of symmetry of the liquid wedge as the bottom, we make a transformation to the plane z: $z = \omega^{1/k}$; $f(z) = z^k$; $k = 1 - \beta/\pi$, where β is the angle between the cheek of the rigid wedge and the negative part of the real axis. The dimensions are $[|\omega|] = L$—length; then $[|z|] = [x] = [y] = [|\omega|^{1/k}] = L^{1/k}$.

We write the equations of motion for this example in the simplest form, retaining only the principal terms. It is obvious that the reducibility of the partial differential equations (6) to ordinary ones in the case of self-similarity does not depend on how many terms in equations (6) we retain:

$$\frac{\partial\varphi}{\partial t} = -\frac{1}{2}\frac{|z|^{2-2k}}{k^2}\left(\frac{\partial\varphi}{\partial x}\right)^2, \qquad \frac{\partial y}{\partial t} = -\frac{|z|^{2-2k}}{k^2}\left[y\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial\varphi}{\partial x}\frac{\partial y}{\partial x}\right]. \tag{6'}$$

We pass in these equations to dimensionless variables by the formulas

$$\lambda = xV_0^{-1/k}t^{-1/k}; \qquad \varphi = V_0^2 t\,\Phi(x); \qquad y = V_0^{1/k}t^{1/k}Y(\lambda),$$

$$|z| = \sqrt{x^2 + Y^2} = V_0^{1/k}t^{1/k}\sqrt{\lambda^2 + Y^2(\lambda)},$$

where $V = CV_0$; V_0 is a constant of the dimension of velocity.

We obtain the following system of ordinary differential equations:

$$k\Phi - \lambda\Phi = -\left|\frac{1}{2kz^{2k-2}}\right|(\Phi')^2; \quad Y - \lambda Y' = -\left|\frac{1}{kz^{2k-2}}\right|[Y\Phi'' + Y'\Phi']. \tag{6"}$$

Differentiating the first equation with respect to λ and then eliminating Φ'' from both equations, we obtain, instead of the second equation,

$$Y' = Y\frac{\frac{2-k}{k}(\lambda^2 + Y^2)^{1-k}\Phi' - \lambda - 4(1-k)(\lambda^2+Y^2)^{2k-1}\lambda(\Phi')^2}{-4\left[\frac{1}{2k}(\lambda^2+Y^2)^{1-k}\Phi' - \frac{1}{2}\lambda\right]^2 + Y^2 4(1-k)(\lambda^2+Y^2)^{2k-1}(\Phi')^2}. \tag{7}$$

Solving the first equation of the system (6) with respect to Φ', we obtain the equation

$$\Phi' = k\frac{\lambda + \sqrt{\lambda^2 - 2(\lambda^2+Y^2)^{1-k}\Phi}}{(\lambda^2+Y^2)^{1-k}} = k\frac{\lambda - \sqrt{\lambda^2 - 2(\lambda^2+Y^2)^{1-k}\Phi}}{(\lambda^2+Y^2)^{1-k}}. \tag{8}$$

The initial data for equations (7) and (8) are the conditions at infinity

Fig. 2

Figure 2: Fig. 2

$$-V = CV_0 = \left(\frac{\partial \varphi}{\partial \operatorname{Re}\omega}\right)_{\omega\to\infty} = \left|\frac{1}{k}2^{1-k}\right|\frac{\partial \varphi}{\partial x} = \frac{V_0}{k}(\lambda^2+Y^2)^{\frac{1-k}{2}}\Phi' \quad \text{for } \lambda=\infty;$$

i.e.,

$$(\Phi')_{\lambda=\infty} = Ck(\lambda^2+Y^2)^{\frac{k-1}{2}}.$$

The second condition is

$$\frac{\partial y}{\partial \operatorname{Re}\omega} = \operatorname{tg}\alpha \quad \text{as } \omega\to\infty.$$

In the z-plane this condition is written in the form

$$\frac{\partial y}{\partial x} = y'_{\lambda=\infty} = \operatorname{tg}^{\frac{1}{k}}\alpha = C_1.$$

It is not difficult to verify that equations (7), (8) admit the required asymptotics for $0 < k < 3/4$. A trivial solution of equation (7) is $Y = 0$. Consequently, if we require that $y(\lambda_1) = 0$, then λ_1 can only be a singular point of equation (7). Such a point will be the point where $(\lambda^2+Y^2)^{1-k}\Phi' - k\lambda = 0$ and $Y = 0$. From (7) and (8) we obtain that, for

$$1/2 < k < 3/4, \quad -\left(\frac{1}{4k^2}\right)^{\frac{1}{4k-2}} < \lambda,$$

and small Y, the numerator of (8) vanishes only at the singular point λ_1, which is a node. We obtain the qualitative picture of the behavior of the integral curves shown in Fig. 2.

Fig. 2

Each integral curve gives the shape of the free surface in the z-plane, and, since we know the conformal relation between the ω- and z-planes, we can find the shape of the free surface in the physical plane.

In conclusion, we note that, since the formula of M. A. Lavrent' ev that we have used admits refinement, equations (4) and (5) also admit refinement. Moreover, they can be generalized to flows with a curvilinear bottom.

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Note: Figure translations are in progress. See original paper for figures.

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