MATHEMATICS

Yu. I. ZHURAVLEV

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Abstract

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Yu. I. ZHURAVLEV

ON THE SEPARABILITY OF SUBSETS OF VERTICES OF THE n-DIMENSIONAL UNIT CUBE

(Presented by Academician M. V. Keldysh, 13 X 1956)

In a number of applied questions there arise functions $F(x_1, \dots, x_n)$, defined on some subset of the vertices of the n-dimensional unit cube and taking the values 0 and 1 (^{1}). In this connection, in the sense of the problem, all possible completions of the given function are allowed within the class of functions of the algebra of logic. In the present work a complete solution is given to the problem of finding such extensions for which the obtained function admits the simplest (in the sense of the number of letters of variables) disjunctive or conjunctive normal form. In the course of solving this problem, a criterion is obtained that makes it possible, in a disjunctive normal form (d.n.f.), to exclude superfluous, absorbed terms. In the last part the question of simplifying a d.n.f. by introducing new variables is considered.

Functions of the algebra of logic depending on n arguments may be regarded as functions defined on the set E_n of all vertices of the n-dimensional unit cube and taking the values 0 and 1. Denote by M_f the subset of vertices of E_n for which $f(x_1, \dots, x_n) = 1$.

Definition. The subset $M_{\mathfrak{A}} \subseteq E_n$, corresponding to the elementary conjunction $*$ $\mathfrak{A}$ of rank k, is called an **interval of rank** k (^{2}).

Let $I = \{M_{\mathfrak{A}}\}$ be some subset of intervals from E_n.

Definition. An interval $M_{\mathfrak{B}} \in I$ is called **maximal relative to** I if there is no interval $M_{\mathfrak{A}}$ in I such that $M_{\mathfrak{B}} \neq M_{\mathfrak{A}}$ and $M_{\mathfrak{A}} \supseteq M_{\mathfrak{B}}$.

Consider a set $M_f \subseteq E_n$ on which $f(x_1, \dots, x_n)$ is equal to 1. Choose all maximal intervals $M_{\mathfrak{B}_i} \subseteq M_f$ $(i = 1, 2, \dots, l)$ and form from the corresponding elementary conjunctions the d.n.f.

$$\bigvee_{i=1}^{l} \mathfrak{B}_i.$$

The obtained d.n.f. is called the **reduced normal form** for $f(x_1, \dots, x_n)$ (^{2})**.

Let $F(x_1, \dots, x_n)$ be given on a subset $M \subseteq E_n$ and take the values 0 and 1. It is obvious that there exist subsets M_1 and M_2 such that $M_1 \cup M_2 = M$,

$M_1 \cap M_2 = 0$, and $F(x_1, \ldots, x_n) = 1$ on $M_1 \subseteq E_n$; $F(x_1, \ldots, x_n) = 0$ on $M_2 \subseteq E_n$. It is obvious that $F(x_1, \ldots, x_n)$ is completely determined by specifying the pair of nonintersecting subsets M_1 and M_2, or by a pair of functions f_1 and f_2 such that $f_1 \& f_2 = 0$ and $M_{f_1} = M_1$, $M_{f_2} = M_2$.

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* An elementary conjunction of rank k is the logical product $\mathfrak{A} = x_{i_1}^{\sigma_1} \cdot x_{i_2}^{\sigma_2} \cdots x_{i_k}^{\sigma_k}$, where all x_{i_j} are distinct and $x^\sigma = x$ for $\sigma = 1$, $x^\sigma = \bar{x}$ for $\sigma = 0$.

** The theory of functions of the algebra of logic, including the concept of an "interval," the new definition of reduced d.n.f., etc., was presented by S. V. Yablonskii in 1955–1956 in lectures and seminars at the Mechanics and Mathematics Faculty of Moscow State University.

There exist various completions of $F(x_1, \ldots, x_n)$ in the class of functions of the algebra of logic that are not equivalent to one another. Consider the class of sets M corresponding to various completions of the function $F(x_1, \ldots, x_n)$, i.e., such that for $M_{f_i} \in M$ the conditions $M_{f_i} \cap M_2 = 0$, $M_{f_i} \supset M_1$ hold. Our problem will be to find completions that are simplest in a certain sense.

Let us, according to some rule, assign to each subset $M_f \subseteq E_n$ a nonnegative integer and call it the simplicity index of the subset M_f and of the function $f(x_1, \ldots, x_n)$. We can now formulate the problem as follows: in the class of sets M, find a set M_f with minimal simplicity index.

In what follows this problem is solved for a special definition of the simplicity index.

Definition. A minimal d.n.f. of a function $f(x_1, \ldots, x_n)$ is called a d.n.f. realizing $f(x_1, \ldots, x_n)$ and having the minimal number of letters.

We define the **simplicity index** of M_f as the number of letters in a minimal d.n.f. for $f(x_1, \ldots, x_n)$. A set M_f with minimal simplicity index has the properties $M_f \supset M_1$, $E_n \setminus M_f \supset M_2$. Thus M_f and $E_n \setminus M_f$ separate M_1 and M_2 and are the simplest separators in our sense. Therefore it is natural to call the problem of finding M_f the problem of logical separability.

Solution of the problem of logical separability. We shall give a geometric and an analytic solution. The first is of interest for theoretical investigations; the second is also of interest from the practical point of view.

Geometric solution. Let $F(x_1, \ldots, x_n)$ be given by a pair of disjoint subsets M_1 and M_2. Select all maximal intervals $M_{\mathfrak{B}_i}$ $(i = 1, 2, \ldots, l)$ such that $M_{\mathfrak{B}_i} \subseteq E_n \setminus M_2$ and having a nonempty intersection with M_1. The d.n.f.

$$\mathfrak{N} = \bigvee_{i=1}^{l} \mathfrak{B}_i$$

is called the **reduced normal form** for $F(x_1, \dots, x_n)$. $\mathfrak{N}$ is uniquely determined by the specification of the sets M_1 and M_2. If $M_1 \cup M_2 = E_n$, then this definition coincides with the one introduced earlier.

Theorem 1. *The minimal d.n.f.* $\mathfrak{N}_{\min}$*, corresponding to the simplest separator* M_f *of the sets* M_1 *and* M_2*, is obtained from the reduced d.n.f.* $\mathfrak{N}$ *for the function* $F(x_1, \dots, x_n)$ *by deleting certain elementary conjunctions* $\mathfrak{B}_i$*.*

Let us note that if the d.n.f.

$$\mathfrak{N} = \bigvee_{i=1}^{s} \mathfrak{A}_i$$

corresponds to a separator of the sets M_1 and M_2, and $\mathfrak{A}_i$ is an elementary conjunction in $\mathfrak{N}$ such that

$$M_1 \cap M_{\mathfrak{A}_i} \subseteq \bigcup_{j \neq i} M_{\mathfrak{A}_j},$$

then the d.n.f.

$$\mathfrak{N}' = \bigvee_{j \neq i} \mathfrak{A}_j$$

also corresponds to a separator of the sets M_1 and M_2 and has a smaller number of letters. Using this observation, we can successively cross out elementary conjunctions from the reduced d.n.f. for $F(x_1, \dots, x_n)$. Trying all possible ways of deleting, we find all $\mathfrak{N}_{\min}$.

Analytic solution. Let there be given

$$f_1(x_1, \dots, x_n) = \bigvee_{i=1}^{l} x_{i_1}^{\sigma_1} x_{i_2}^{\sigma_2} \cdots x_{i_n}^{\sigma_n} = \bigvee_{i=1}^{l} \mathfrak{A}_i,$$

$$f_2(x_1, \dots, x_n) = \bigvee x_{j_1}^{\sigma_1} x_{j_2}^{\sigma_2} \cdots x_{j_s}^{\sigma_s}$$

such that $M_1 = M_{f_1}$, $M_2 = M_{f_2}$. Obviously $f_1 \cdot f_2 = 0$,

We divide the analytic solution into stages:

1. Selection of all maximal intervals wholly contained in $E_n \setminus M_2$. The construction is based on the relation $E_n \setminus M_2 = M_{\bar{f}_2}$ and can be carried out by Nelson' s method ([3]).
2. Selection of the maximal intervals from $E_n \setminus M_2$ having a nonempty intersection with M_1.

3. Deletion from the reduced d.n.f. for $F(x_1, \dots, x_n)$ of the elementary conjunctions corresponding to intervals absorbed inside the set M_1 by the sum of the remaining intervals. This stage may be split into elementary steps, each of which consists in deleting from the d.n.f. obtained at the preceding step one elementary conjunction. The latter is established with the aid of an analytic criterion which determines the case in which a part of some interval $M_{\mathfrak{A}_0}$, contained in the given set $M_{\mathfrak{A}}$, is covered by the sum of intervals

$$\bigcup_{i=1}^{k} M_{\mathfrak{A}_i},$$

where

$$\mathfrak{A} = \bigvee_{i=1}^{l} \mathfrak{A}_i.$$

With these designations our problem consists in finding conditions under which the relation

$$\left(\mathfrak{C}_0 \cdot \mathfrak{A} \to \bigvee_{i=1}^{k} \mathfrak{C}_i\right) \equiv 1$$

holds.

Let

$$\bigvee_{j=1}^{s} \mathfrak{C}'_j$$

be formed from all conjunctions $\mathfrak{C}_i$ $(i = 1, 2, \dots, s)$ not orthogonal to $\mathfrak{C}_0$. Similarly,

$$\mathfrak{A}' = \bigvee_{i=1}^{r} \mathfrak{A}'_i$$

is constructed from all conjunctions $\mathfrak{A}_i$ not orthogonal to $\mathfrak{C}_0$. Obviously, the condition

$$\left(\mathfrak{C}_0 \cdot \mathfrak{A} \to \bigvee_{j=1}^{k} \mathfrak{C}_i\right) \equiv 1$$

is equivalent to the condition

$$\left(\mathfrak{C}_0 \cdot \mathfrak{A}' \to \bigvee_{j=1}^{s} \mathfrak{C}'_j\right) \equiv 1.$$

Theorem 2.

$$\left(\mathfrak{A}' \cdot \mathfrak{C}_0 \to \bigvee_{i=1}^{s} \mathfrak{C}'_j\right) \equiv 1$$

if and only if

$$\mathfrak{C}'_j = \mathfrak{C}_j \cdot v_j \quad (j = 1, 2, \dots, s),$$

where $\mathfrak{C}_j, v_j$ are elementary conjunctions, and

$$\left(\mathfrak{A}' \to \bigvee_{j=1}^{s} \mathfrak{C}_j\right) \equiv 1 \qquad \left(\mathfrak{C}_0 \to \bigvee_{j=1}^{s} v_j\right) \equiv 1.$$

On the choice of essential variables. The cumbersomeness of the solution of the separability problem depends essentially on the number of variables of the function $F(x_1, \dots, x_n)$. We shall discuss the possibility of constructing the simplest separator both by introducing new variables and by reducing the number of existing variables.

Definition. It is said that a function $\Phi(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+k})$ is obtained from a function $F(x_1, \dots, x_n)$ by introducing new variables $x_{n+1}, \dots, x_{n+k}$, if for every tuple $\alpha_1, \dots, \alpha_n$ for which the function $F(x_1, x_2, \dots, x_n)$ is defined, there exist numbers $\alpha_{n+1}, \dots, \alpha_{n+k}$ $(0 \le \alpha_{n+1}, \dots, \alpha_{n+k} \le 1)$ such that

$$\Phi(\alpha_1, \dots, \alpha_n, \alpha_{n+1}, \dots, \alpha_{n+k}) = F(\alpha_1, \dots, \alpha_n).$$

It is possible to obtain simpler separators by introducing new variables. Let $f(x_1, \dots, x_n)$ be a function of the algebra of logic. Obviously,

$$\Phi(x_1, \dots, x_n, x_{n+1}) \equiv x_{n+1}$$

is obtained from the function $f(x_1, \dots, x_n)$ by introducing the variable x_{n+1}. This example shows that, when new-

of the variable x_{n+1}, the values of the function $\Phi(x_1, \dots, x_n, x_{n+1})$ on the tuples $\alpha_1, \dots, \alpha_n, 0$ and $\alpha_1, \dots, \alpha_n, 1$ are, generally speaking, different and are not determined by the value of the function $F(x_1, \dots, x_n)$ on the tuple $\alpha_1, \dots, \alpha_n$. Therefore the concept of "introducing new variables" needs clarification.

Definition. We say that the variables $x_{n+1}, \dots, x_{n+k}$ are introduced into the function $F(x_1, \dots, x_n)$ in a natural way if, for the tuples $(\alpha_1, \dots, \alpha_n)$ on which the function $\Phi(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+k})$ is defined, the following relation holds between the functions $\Phi(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+k})$ and $F(x_1, \dots, x_n)$:

$$\Phi(\alpha_1, \dots, \alpha_n, x_{n+1}, \dots, x_{n+k}) \equiv F(\alpha_1, \dots, \alpha_n).$$

Theorem 3. *Under a natural introduction of new variables $x_{n+1}, \dots, x_{n+k}$ (where k is an arbitrary positive number) into the function $F(x_1, \dots, x_n)$, the simplest separator for the function $\Phi(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+k})$ obtained by this completion has a simplicity index no smaller than that of the simplest separator of the function $F(x_1, \dots, x_n)$.*

For selecting sets of variables $x_{i_1}, \dots, x_{i_k}$ from the set of all variables $x_1, \dots, x_n$ $(k \leq n)$, through which logical separators for the function $F(x_1, \dots, x_n)$ are expressed, the following method is possible.

Let $F(x_1, \dots, x_n)$ be given by a table. We subtract, componentwise, the tuples for which $F = 0$ from the tuples for which $F = 1$. We select the variables in whose positions zeros are obtained, and form from them all possible sets. All sets of variables that cannot be obtained in this way are the required ones.

Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.