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Abstract

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REPRESENTATION OF SOLUTIONS OF THE EULER-POISSON-DARBOUX EQUATION BY ANALYTIC FUNCTIONS

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In the present article, for the equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{c}{y}\frac{\partial w}{\partial y} = 0, \qquad c = \text{const}, \tag{1}$$

for values of c belonging to the interval $0 < c < 1$, the results set forth in the article ([1]) are strengthened. In the notation adopted there, for any domain T adjoining an interval L of the axis Ox, the following theorem is proved.

Theorem 1. *If a solution $w(x, y)$ of equation (1), belonging to the class $C_2(T)$, or the limit*

$$\lim_{y\to 0} y^c \frac{\partial w}{\partial y} \tag{2}$$

takes analytic values in x on the interval L, then there exists a domain σ, adjoining L, in which the solution can be represented in the form

$$w(x, y) = \gamma\left(\frac{c}{2}\right)\int_0^1 \frac{\varphi[x + iy(1 - 2\sigma)]\, d\sigma}{|\sigma(1-\sigma)|^{1-c/2}} +$$

$$+\gamma\left(1 - \frac{c}{2}\right)\left(\frac{y}{1-c}\right)^{1-c}\int_0^1 \frac{\psi[x + iy(1-2\sigma)]\, d\sigma}{[\sigma(1-\sigma)]^{c/2}}. \tag{3}$$

In this expression $\varphi(z)$ and $\psi(z)$ are functions analytic in $\sigma \cup L \cup \bar{\sigma}$, satisfying on L the conditions

$$\varphi(x) = w(x, 0). \tag{4}$$

$$\psi(x) = \lim_{y\to 0}\left(\frac{y}{1-c}\right)^c \frac{\partial w}{\partial y}. \tag{5}$$

Proof. In the case when the solution is analytic on L, the function $\varphi(x) = w(x,0)$ is analytically continued to complex values, forming in some domain $\sigma' \cup L \cup \bar{\sigma}'$ an analytic function $\varphi(z)$. In σ' consider the open semicircle τ of radius ρ, adjoining the interval ab (or l) of the axis Ox and bounded, for $y > 0$, by the semicircle γ.

Using the values of $w(x,y)$ on the boundary of the semicircle $\gamma \cup l \cup \{a,b\}$, we solve problem D in τ ([2]). We seek its solution in the form of the sum $w_1(x,y) + w_2(x,y)$, where $w_1(x,y)$ has the form of the first term of expression (3), in which $\varphi(z)$ is just the analytic function considered.

We note the following property of solutions of equation (1). If $w(x,y)$ satisfies equation (1) for $c = 2-2\beta$ and $y > 0$, then the expression $y^{1-2\beta}w(x,y)$ will also satisfy (1) for $c = 2\beta$.

We seek $w_2(x,y)$ in the form $y^{1-2\beta}w_2^*(x,y)$, where $w_2^*(x,y)$ is a solution of equation (1) for $c = 2-2\beta$, belonging to the class $C_2(\tau)$, continuous on γ, and taking on γ the values

$$\left(\frac{y}{1-2\beta}\right)^{2\beta-1}[w(x,y) - w_1(x,y)]. \tag{6}$$

If on γ we introduce the variable $t = \dfrac{x - x_0}{\rho}$, then expression (6) on γ will be a function $F(t)$, defined on the interval $(-1,1)$. Generally speaking, on this interval $F(t)$ is unbounded, but $(1-t^2)^{1/2-\beta}F(t)$ is always bounded. Therefore, on the basis of work ([3]), $F(t)$ can be expanded in the series

$$\sum_{n=0}^{\infty} a_n C_n^{1-\beta}(t),$$

converging uniformly on every segment $[-1+\delta, 1-\delta]$ $(0 < \delta < 1)$.

To construct $w_2^*(x,y)$ in τ, we apply the method used in work ([1]). We obtain an expression for $w_2^*(x,y)$ in τ in the form

$$w_2^*(x,y) = \gamma\left(1 - \frac{c}{2}\right)\int_0^1 \frac{\psi[x + iy(1-2\sigma)]\,d\sigma}{[\sigma(1-\sigma)]^{c/2}}, \tag{7}$$

where $\psi(z)$ is a function analytic in $\tau \cup l \cup \overline{\tau}$, real on l. Moreover, we find that $w_2^*(x,y)$ is continuous on γ and takes on γ the values (6), while the expression $y^{1-2\beta}w_2^*(x,y)$ is bounded in τ. Consequently, the constructed solution $w^*(x,y) = w_1(x,y) + w_2(x,y)$ in the form (3), where $\varphi(z)$ and $\psi(z)$ are functions analytic in $\tau \cup l \cup \overline{\tau}$, belongs to the class $C_2(T)$, is continuous on γ, and on $l \cup \gamma$ takes the values $w(x,y)$. And since $w^*(x,y)$ is bounded in τ, it is proved by the usual barrier method ([2]) $(v = -\ln(x^2+y^2))$ that $w(x,y) \equiv w^*(x,y)$ in τ, i.e. $w(x, y)$ is representable in τ in the form (3). Since every point of the interval l is an interior point of the domain of definition of the functions $\varphi(z), \psi(z)$, and on l the functions $\varphi(z), \psi(z)$ satisfy the relations (4), (5), it follows, by the uniqueness property of analytic functions, that $\varphi(z)$ and $\psi(z)$ in the expression (3) are unique. The latter means that the representation of $w(x, y)$ in the form (3) is unique. Considering the representations of $w(x, y)$ in the collection of semicircles τ_n adjacent to L, we obtain the representation of $w(x, y)$ in the form (3) in a certain domain σ adjacent to L.

The second case of the theorem is proved analogously.

Corollary. If the solution $w(x, y)$ is analytic in x on L, then the limit (7) is analytic in x on L, and conversely.

Two functions $w(x, y)$ and $w^*(x, y)$, defined in T, will be called **conjugate** if they belong to the class $C_2(T)$ and on L satisfy the relations

$$w(x, 0) = w^*(x, 0), \qquad \lim_{y\to 0} y^c \frac{\partial w}{\partial y} = -\lim_{y\to 0} y^c \frac{\partial w^*}{\partial y}.$$

Theorem 2. For the representability of a solution $w(x, y)$ of equation (1), for $0 < c < 1$, belonging to the class $C_2(T)$, where $T \in B$, in the form (3), it is necessary and sufficient that at least one of the following conditions be fulfilled:

a) the existence in T of a conjugate solution;

b) the existence of a function analytic in T satisfying condition (4) on the interval L;

c) the existence of a function analytic in T and satisfying condition (5) on the interval L.

We shall first carry out the proof for case b) of the theorem. In this case, by the preceding, there exists a domain σ adjoining the interval L in which the solution is represented in the form (3), where $\varphi(z)$ is a function analytic in $T \cup L \cup \overline{T}$, and $\psi(z)$ is analytic in $\sigma \cup L \cup \overline{\sigma}$. From this representation it is clear that the solution $w(x, y)$ extends to complex values of z and ζ in the form

$$U(z, \zeta) = w\left(\frac{z+\zeta}{2}; \frac{z-\zeta}{2i}\right) \tag{8}$$

to the domain

$$\{z \in \sigma \cup L \cup \overline{\sigma},\ \zeta \in \sigma \cup L \cup \overline{\sigma},\ \operatorname{Im} z > \operatorname{Im} \zeta\}$$

or to the domain $Q(\sigma)$. Moreover, the real solution $w(x, y)$ defined in T extends, by a theorem of I. N. Vekua (4), to complex values of z and ζ in the form (8) in the bicylindrical domain $\{z \in T,\ \zeta \in T\}$, or in the domain $B(T)$. Therefore $w(x, y)$ determines an analytic function $U(z, \zeta)$ in the sum of the domains $Q(\sigma)$ and $B(T)$.

The expression (3), extended to complex values of z and ζ in the form (8), may be written as

$$\gamma(1-\beta)\int_0^1 \frac{\psi[z+(\zeta-z)\sigma]\,d\sigma}{[\sigma(1-\sigma)]^{\beta}} = \frac{U(z,\zeta)-\gamma(\beta)\displaystyle\int_0^1 \frac{\varphi[z+(\zeta-z)\sigma]\,d\sigma}{[\sigma(1-\sigma)]^{1-\beta}}}{\left[\dfrac{z-\zeta}{2i(1-2\beta)}\right]^{1-2\beta}}. \quad (9)$$

We choose that branch of the expression in the denominator of the right-hand side which is real and positive for real and positive $i(\zeta - z)$.

Equality (9) is valid in the domain $Q(\sigma)$, but its left-hand side is defined in $B(\sigma \cup L \cup \overline{\sigma})$, and the right-hand side in the sum of $Q(\sigma)$ and $B(T)$. Since the right-hand and left-hand sides separately represent analytic functions of two complex variables z and ζ in the corresponding domains and coincide in the bicylindrical domain $B(\sigma)$, they both represent one analytic function of two complex variables $V(z,\zeta)$, defined in the sum $B(\sigma \cup L \cup \overline{\sigma})$ and $B(T)$. For values $\zeta = \overline{z}$ this function $V(z,\overline{z})$ is a real solution of equation (1) with $c_1 = 2-c > 1$ of class $C_2(T)$, and therefore, by (*), is represented in T in the form of the left-hand side of expression (9), in which $\psi(z)$ is a function analytic in the domain $T \cup L \cup \overline{T}$ and having real values on L. Constructing with its aid the expression (3), we obtain the required representation. The proof of the theorem in case c) is carried out in a similar way.

In case a) consider the half-sum $w_1(x,y)$ and the half-difference $w_2(x,y)$ of the conjugate solutions. Both belong to the class $C_2(T)$ and have the following properties: the half-sum $w_1(x,y)$ on L has the zero boundary value (5), while the half-difference $w_2(x,y)$ is equal to zero on L, i.e. $w_2(x,0) = 0$. Therefore, in accordance with conditions c) and b) of Theorem 2, they are represented in T in the form (3), where $\psi(z) \equiv 0$ in the first case and $\varphi(z) \equiv 0$ in the second. Hence the solution itself

$$w(x,y) = w_1(x,y) + w_2(x,y)$$

is represented in the form (3). According to the remark to Theorem 1, the representation is unique.

Corollary. Every solution of class $C_2(T)$ of equation (1) for $0 < c < 1$, representable in T in the form (3), analytically extends

in the form (8) to the following domain of two complex variables z and ζ:

$$\left\{ \begin{array}{c} z \in D \\ \zeta \in D \\ \operatorname{Im} z > \operatorname{Im}\zeta \end{array} \right\}, \qquad D = T \cup L \cup \overline{T}.$$

In the domain $D = T \cup L \cup \overline{T}$, where $T \in B$, consider the class of functions $C_2(D)$. To it we assign every pair of functions belonging, respectively, to the classes $C_2(T)$ and $C_2(\overline{T})$ and forming in D, together with the expression $|y|^c \dfrac{\partial w}{\partial y}$, a single continuous function.

Theorem 3. Every solution $w(x, y)$ of class $C_2(D)$ of equation (1), for $0 < c < 1$, is represented in D in the form

$$w(x,y) = \gamma\left(\frac{c}{2}\right)\int_0^1 \frac{\varphi[x+iy(1-2\sigma)]\,d\sigma}{[\sigma(1-\sigma)]^{1-c/2}} +$$

$$+\operatorname{sign} y\cdot\gamma\left(1-\frac{c}{2}\right)\left(\frac{|y|}{1-c}\right)^{1-c}\int_0^1 \frac{\psi[x+iy(1-2\sigma)]\,d\sigma}{[\sigma(1-\sigma)]^{c/2}}, \tag{10}$$

where $\varphi(z)$ and $\psi(z)$ are functions analytic in D, satisfying on L, respectively, condition (4) and the condition

$$\psi(z) = \lim_{y\to 0}\left(\frac{|y|}{1-c}\right)^c \frac{\partial w}{\partial y}. \tag{11}$$

Proof. Since, according to the definition of the class $C_2(D)$, the solution $w(x, y)$ consists of two functions $w^1(x, y)$ and $w^2(x, y)$, defined, respectively, in T and $\overline{T}$, and since equation (1) does not change under the replacement of y by $-y$, the function $\overline{w(x, y)} = w^2(x, -y)$, defined in T, is a solution of equation (1) for $0 < c < 1$ and has on L the properties

$$w^1(x,0) = \overline{w(x,0)}, \qquad \lim_{y\to 0} y^c \frac{\partial w^1}{\partial y} = -\lim_{y\to 0} y^c \frac{\partial \overline{w}}{\partial y},$$

i.e. $\overline{w(x, y)}$ is the solution of equation (1) in T conjugate to $w^1(x, y)$.

Consequently, $w(x, y)$ is represented in T in the form (10), where $\varphi(z)$ and $\psi(z)$ are functions analytic in D and taking real values on L. Representing $w(x, y)$ by this method also in the domain $\overline{T}$, we obtain a representation of $w(x, y)$ in the domain $D = T \cup L \cup \overline{T}$, which in both cases is written in the form (10). By the preceding result the representation is unique, as was required to prove.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.