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Abstract

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APPLICATION OF THE LONG-WAVE METHOD TO THE PROBLEM OF FLOW OVER MOUNTAINS

(Presented by Academician A. A. Dorodnitsyn, 19 II 1957)

Let us consider, in a coordinate system x, z (x is the horizontal coordinate, z the vertical coordinate directed upward), a plane steady nonlinear problem of an air flow over a mountain whose horizontal dimensions are so much greater than the vertical ones that the long-wave method may be used to solve the problem. The idea of applying this method to the present problem was advanced by I. A. Kibel' ([1]), but I. A. Kibel' confined himself to reducing the problem to a very complicated ordinary nonlinear differential equation*. A general solution of this equation has not yet been obtained; only certain results have been obtained in a greatly simplified formulation of the problem ([2]).

With the aim of estimating the influence of the vertical velocity gradient on the character of the flow, in the present paper we shall consider the case in which the velocity of the oncoming flow far from the mountain on the windward side varies with height according to a linear law. Starting from the general system of equations of thermohydrodynamics of the atmosphere, we shall simplify the theory of convection and, in addition, neglect turbulence and the Coriolis force.

Then we arrive at the following system of equations:

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -R\theta\frac{\partial}{\partial x}\left(\frac{p'}{P}\right); \qquad -R\theta\frac{\partial}{\partial z}\left(\frac{p'}{P}\right) = \lambda\vartheta \quad \left(\lambda = \frac{g}{\theta}\right);$$

$$u\frac{\partial \vartheta}{\partial x} + w\frac{\partial \vartheta}{\partial z} + \mu w = 0; \qquad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = \sigma w \quad \left(\sigma = \frac{g - R\gamma}{R\theta}\right). \tag{1}$$

Here u, w are the components of the wind velocity along the axes x, z, respectively; ϑ, p' are the deviations of temperature and pressure from the values θ and P of these quantities in the undisturbed flow; R is the gas constant for air; $\mu = \gamma_a + d\theta/dz$, where γ_a is the dry-adiabatic gradient. We shall assume the atmosphere to be stably stratified**; in other words, we require that $\mu > 0$.

Let us place the origin of coordinates on the earth' s surface near the base of the mountain on the windward side and assume that at the height H, which we shall regard as known, there is a solid smooth horizontal wall bounding the flow from above***. Then, in accordance with the formulation of the problem, the boundary—

* A. A. Dorodnitsyn succeeded in obtaining, by a numerical method, a particular solution of this equation (report at a seminar of the Department of Dynamic Meteorology of the Central Institute of Forecasts, 1949). At the same time it was established for the first time that closed streamlines can arise in the atmosphere above an obstacle being flowed around. Similar phenomena have recently been discovered by observations in nature (Küttner and others).

** The problem admits a solution also in the case $\mu \leq 0$, but for meteorology this case is not of great interest.

*** It has been shown ([3]) that, for the present problem, the assumption of the presence of such a wall is practically identical with the assumption of the existence of a free surface of the flow.

···boundary conditions can be written in the form

$$u = a + bz, \quad \vartheta = 0 \quad \text{for } x = -\infty; \tag{2}$$

$$w = u\delta'(x) \quad \text{for } z = \delta(x) \geq 0; \qquad w = p' = 0 \quad \text{for } z = H,$$

where $a > 0$ and $b \geq 0$ are prescribed constants; $z = \delta(x)$ is the equation of the mountain profile.

We now proceed to the solution of the problem, first setting $\mu = \text{const}$, $\theta = \text{const}$, $\sigma = 0$.

Introduce into (1) and (2) the stream function ψ from the relations $u = \partial\psi/\partial z$; $w = -\partial\psi/\partial x$, and then eliminate the pressure from system (1). Then, instead of (1) and (2), we obtain

$$\frac{\partial\psi}{\partial x}\frac{\partial\vartheta}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial\vartheta}{\partial x} + \mu\frac{\partial\psi}{\partial x} = 0; \tag{3}$$

$$\frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial z^3} - \frac{\partial\psi}{\partial z}\frac{\partial^3\psi}{\partial x\partial z^2} - \lambda\frac{\partial\vartheta}{\partial x} = 0; \tag{4}$$

$$\psi = az + \frac{b}{2}z^2 = \psi_\infty(z); \qquad \vartheta = 0 \quad \text{for } x = -\infty; \tag{5}$$

$$\psi = 0 \quad \text{for } z = \delta(x); \qquad \psi = aH + \frac{b}{2}H^2 \quad \text{for } z = H. \tag{6}$$

After ϑ has been found, the pressure disturbance p' can be computed by quadrature. Therefore, in what follows the equation for p' is not considered.

In work (5), which concerns the theory of convection, it is indicated that equations (3), (4) have first integrals*

$$\vartheta = f_1(\psi) - \mu z; \tag{7}$$

$$\frac{\partial^2 \psi}{\partial z^2} = f_2(\psi) + \lambda z f_1'(\psi), \tag{8}$$

where $f_1(\psi)$ and $f_2(\psi)$ are arbitrary functions. In the present problem the form of these functions is determined by means of the conditions (5).

Indeed, putting $x = -\infty$ in (7), in consequence of (5) we have

$$f_1(\psi_\infty) = \mu z, \qquad \text{where } z = (\sqrt{a^2 + 2b\psi_\infty} - a)/b.$$

Replacing here the argument ψ_∞ by ψ, which is permissible since the form of the function is not thereby changed, we obtain

$$f_1(\psi) = (\sqrt{a^2 + 2b\psi} - a)\mu/b.$$

In an analogous way, from (8) we find $f_2(\psi)$. Substituting $f_1(\psi)$ and $f_2(\psi)$ into (8), we shall have for ψ the nonlinear equation**

$$\frac{\partial^2 \psi}{\partial z^2} = b - \frac{\mu\lambda}{b} + \frac{\mu\lambda(a + bz)}{b\sqrt{a^2 + 2b\psi}}. \tag{9}$$

* Equation (3) can be represented in the form $\partial(\psi, \vartheta + \mu z)/\partial(x, z) = 0$. Hence follows the existence of the integral (7). In an analogous way one can verify the validity of (8).

** Passing in (9) to the limit $b \to 0$, we obtain a linear equation, the solution of which was studied in detail by R. R. Long (4). We also note that, if the complete third equation of motion were used rather than the equation of statics, as is done in agreement with the long-wave method, then in the left-hand side of equation (9) instead of $\partial^2\psi/\partial z^2$ we would have the Laplacian of ψ with respect to the variables x, z.

We now introduce the dimensionless quantities Z, Δ, α, and β from the relations $Z = z/H$, $\Delta = \delta/H$, $\alpha = a/bH$, $\beta = b/\sqrt{\mu\lambda - 0.25b^2}$, and pass in (9) to the new independent dimensionless variable

$$y = \frac{a}{\alpha\beta} \int\limits_Z^1 \frac{dZ}{\sqrt{a^2 + 2b\psi}}, \tag{10}$$

taking Z as the unknown function. As a result, instead of (9) we shall have

$$Z'' + \left(1 - \frac{3}{4}\beta^2\right) Z' + \left(\beta + \frac{1}{4}\beta^3\right)(Z + \alpha) = 0, \tag{11}$$

where a prime denotes differentiation with respect to y. According to (6), equation (11) must be solved subject to the boundary conditions

$$\begin{aligned} &Z = 1, \quad Z' = -(1+\alpha)\beta \quad \text{for } y = 0; \\ &Z = \Delta, \quad Z' = -\alpha\beta \qquad\quad \text{for } y = Y, \end{aligned} \tag{12}$$

where Y is the value taken by y when the lower limit of the integral (10) becomes equal to Δ. It is evident that the characteristic equation corresponding to the linear equation (11) has the root $-\beta$. Knowing one root, we readily find the other two: $0.5\beta + i$.

Thus, the general solution of equation (11) will be

$$Z = c_1 e^{-\beta y} + e^{\frac{1}{2}\beta y}(c_2 \sin y + c \cos y) - \alpha. \tag{13}$$

Substituting (13) into (12), we obtain a system of 4 algebraic equations for the unknowns c_1, c_2, c, and Y. From this system we find $c_1 = 1+\alpha-c$, $c_2 = -1.5\beta c$, and

$$c = c(Y) = \frac{e^{-\frac{3}{2}\beta Y}\left[1 - \alpha\left(e^{\beta Y} - 1\right)\right]}{e^{-\frac{3}{2}\beta Y} - \cos Y - \left(\frac{1}{\beta} + \frac{3}{4}\beta\right)\sin Y}, \tag{14}$$

where Y must be determined from the transcendental equation

$$\Delta = F(Y) = \left(\frac{1}{\beta} + \frac{9}{4}\beta\right) e^{\frac{1}{2}\beta Y} c(Y) \sin Y, \tag{15}$$

which, apparently, is most simply done graphically by plotting the curve $\Delta = F(Y)$. To select from the countless set of solutions of equation (15) only those values of Y that satisfy the problem, we note that for $\Delta = 0$ it is necessary that $c = 0$, as can be verified by substituting Z from (13) and the expression for ψ

$$\psi = \frac{a^2}{2b}\left[\left(\frac{Z'}{\alpha\beta}\right)^2 - 1\right], \tag{16}$$

Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

which is a consequence of (10), into condition (5). Hence it follows that, in seeking Y, one may use only that portion of the curve $\Delta = F(Y) \geq 0$ which adjoins the point $Y = \beta^{-1} \ln(1+\alpha^{-1})$. At this point c and Δ, according to (14), vanish.

It is now not difficult to construct formulas for u, w, ϑ, and p', and all these quantities, like Z and ψ, will depend only on the two variables Δ and y. If y is taken as the independent parameter, the calculation of any function of interest to us encounters no difficulty.

Thus, the posed problem has been solved. Analyzing the solution obtained, we ascertain that there are three distinct regions of values of the parameters α and β (in the plane of the variables α, β).

First, the region in which the solution exists for arbitrary mountain heights. This region is the largest in area and, moreover, it includes those values

···of values of α and β, which most often correspond to the actual distribution of wind in the atmosphere. Therefore, the occurrence in nature of motions close in character to the solutions belonging to this region is most probable.

Fig. 1

Secondly, the region in which the problem has a continuous solution only in the case when the height of the mountain does not exceed a certain critical value, depending on the values of the numbers α and β. (It should be noted that the critical heights for most values of α and β lie within the range 0.5 –1.5 km. The greatest value of the critical height is about 2.5 km.) If the height of the mountain coincides with the critical one*, then, according to the solution of the problem, on the lee side of the mountain there should form a region (sometimes two or three regions, at different heights) of convergence of streamlines (something like a jet flow). In a number of cases the meteorological picture resembles a bora. If the height of the mountain does not reach the critical value, the solution gives no such features. The existence of solutions corresponding to the critical heights of mountains is apparently explained by the occurrence of a kind of nonlinear resonance, caused to a considerable extent by the existence of a vertical velocity gradient in the oncoming flow. Finally, in the third region of values of α and β, very small in area, the problem has no continuous solution at all. It seems to us that this fact indicates the physical necessity, for the given values of α and β, of the formation of vortices and separation of jets from the surface of the mountain.

Fig. 2

The pattern of streamlines, calculated from formulas (13)–(16) for the case of a mountain whose height is equal to the critical height, is shown in Fig. 1. An analogous pattern for a mountain whose height does not reach the critical value is given in Fig. 2.

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CITED LITERATURE

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* If a mountain of critical height has the form of a plateau, the solution turns out to be double-valued.

Note: Figure translations are in progress. See original paper for figures.