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L. V. KANTOROVICH and G. Sh. RUBINSHTEIN

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Abstract

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MATHEMATICS

L. V. KANTOROVICH and G. Sh. RUBINSHTEIN

ON ONE FUNCTIONAL SPACE AND SOME EXTREMAL PROBLEMS

(Presented by Academician A. N. Kolmogorov on 27 March 1957)

The paper considers a linear normed space $\Phi(B)$ of completely additive functions defined on the family B of Borel sets of an arbitrary metric compactum R . In contrast to the usual norm, equal to the total variation of a function, a norm is introduced that essentially uses the metric in R . In this case the space conjugate to $\Phi(B)$ turns out to be the space $\text{Lip}^1(R)$ of functions defined on R and satisfying the Lipschitz condition. The space introduced makes it possible to investigate a number of extremal problems.

1. The space $\Phi_0(B)$. Denote by $\Phi_0(B)$ the set of functions $\varphi \in \Phi(B)$ for which $\varphi(R) = 0$. For $\varphi \in \Phi_0(B)$, the set Ψ_φ of nonnegative functions $\psi(e, e')$, completely additive in each argument, defined on $B \times B$ and satisfying the condition $\psi(e, R) - \psi(R, e) \equiv \varphi(e)$, is nonempty. Put

$$\|\varphi\| = \inf_{\psi \in \Psi_\varphi} \iint_{RR} r(x, y) \psi(de, de'),$$

where $r(x, y)$ is the distance in R , and the integral is understood in the sense of Kolmogorov; moreover, the infimum is always attained. $\Phi_0(B)$ is a linear normed space.

Theorem 1. *The space conjugate to $\Phi_0(B)$ is the space $\text{Lip}^1(R)$ with norm*

$$\|u\|_{\sim} = \sup_{x, y \in R, x \neq y} \frac{|u(x) - u(y)|}{r(x, y)}$$

(functions differing by a constant term are identified).

Proof. To each function $u \in \widetilde{\text{Lip}}^1(R)$ there corresponds the functional

$$L(\varphi) = \int_R u(y) \varphi(de). \quad (1)$$

For any $\psi \in \Psi_\varphi$ we have

$$\begin{aligned}
 L(\varphi) &= \int_R u(x) [\psi(de, R) - \psi(R, de)] = \iint_{RR} u(x) \psi(de, de') - \iint_{RR} u(y) \psi(de, de'), \\
 |L(\varphi)| &= \left| \iint_{RR} [u(x) - u(y)] \psi(de, de') \right| \leq \\
 &\leq \|u\|_\sim \cdot \inf_{\psi \in \Psi_\varphi} \iint_{RR} r(x, y) \psi(de, de') = \|u\|_\sim \cdot \|\varphi\|. \tag{2}
 \end{aligned}$$

Consideration of the functions

$$\varphi_{x_0, y_0}(e) = \begin{cases} 1 & \text{if } x_0 \in e, y_0 \in \bar{e}, \\ -1 & \text{if } x_0 \in \bar{e}, y_0 \in e, \\ 0 & \text{in the remaining cases} \end{cases}$$

shows that the norm of the linear functional L is equal to $\|u\|_\sim$.

If L' is an arbitrary linear functional in $\Phi_0(B)$, then the function

$$u(x) = L'(\varphi_{xy}) \in \widetilde{\text{Lip}}^1(R)$$

$$(|u(x) - u(y)| = |L'(\varphi_{xy}) - L'(\varphi_{yy})| = |L'(\varphi_{xy})| \leq \|L'\| \cdot \|\varphi_{xy}\| = \|L'\| \cdot r(x, y))$$

and the functional (1) corresponding to this function coincides with L' on an everywhere dense set (of linear combinations of the functions φ_{xy}), and therefore everywhere.

Let us now compare the space $\Phi_0(B)$ with the space $\Phi_0^V(B)$, consisting of the same elements but normed in the usual way: $\|\varphi\|_V = \text{Var } \varphi$. First of all,

$$\|\varphi\| = \inf_{\psi \in \Psi_\varphi} \iint_{RR} r(x, y) \psi(de, de') \leq \text{diam } R \cdot \inf_{\psi \in \Psi_\varphi} \psi(R, R) = \text{diam } R \cdot \frac{1}{2} \|\varphi\|_V.$$

On the other hand, if R contains at least one point of condensation, then the two norms under consideration are not equivalent, and consequently $\Phi_0(B)$ is not a complete space. However, if $\{\varphi_n\}$ is a Cauchy sequence in $\Phi_0(B)$ and $\|\varphi_n\|_V$ are bounded, then $\{\varphi_n\}$ converges in the norm of $\Phi_0(B)$ to some $\varphi_0 \in \Phi_0(B)$. In general, convergence of $\{\varphi_n\}$ in $\Phi_0(B)$, under the condition of boundedness in $\Phi_0^V(B)$, is equivalent to the weak convergence of $\{\varphi_n\}$ as functionals in $C(R)$ (for the one-dimensional case compare ⁽¹⁾, p. 153).

2. The space $\Phi(B)$. Let $p(x)$ be some function defined on R and satisfying the conditions

$$p(x) > \sup_{y \in R} r(x, y), \quad |p(x) - p(y)| \leq r(x, y) \quad (3)$$

(in particular, $p(x)$ may be taken equal to some constant $C > \text{diam } R$). For $\varphi \in \Phi(B)$ put

$$\|\varphi\| = \inf_{\varphi_0 \in \Phi_0(B)} \left[\|\varphi_0\| + \int_R p(x) |\varphi(de) - \varphi_0(de)| \right],$$

where $\|\varphi_0\|$ is the norm defined for $\varphi \in \Phi_0(B)$; here the infimum is always attained. $\Phi(B)$ is a linear normed space; $\Phi_0(B)$ is its subspace. The space conjugate to $\Phi(B)$ is the space $\text{Lip}^1(R)$ of functions u with norm

$$\|u\| = \max \left\{ \|u\|_{\sim}, \left\| \frac{u}{p} \right\|_{C(R)} \right\} \quad \left(\|u\|_{C(R)} = \max_{x \in R} |u(x)| \right).$$

Every linear functional in $\Phi(B)$ has the form

$$L(\varphi) = \int_R u(x) \varphi(de), \quad \text{where } u \in \text{Lip}^1(R).$$

3. The problem of mass transportation and some of its generalizations. Let the nonnegative functions $\varphi_1, \varphi_2 \in \Phi(B)$ characterize the first—the available—and the second—the required distribution of mass in R , with $\varphi_1(R) = \varphi_2(R)$. To pass from the available state to the required one it is necessary to carry out a transportation characterized by a function $\psi \in \Psi_\varphi$, where $\varphi = \varphi_1 - \varphi_2$ ($\psi(e, e')$ indicates the amount of mass transported from e to e'). The transportation costs are determined by the integral

$$\iint_{RR} r(x, y) \psi(de, de').$$

A transportation that transforms the available distribution into the required one with minimal costs will be called **optimal**.

Theorem 2*. *For the optimality of a transportation determined by a function $\psi \in \Psi_\varphi$, it is necessary and sufficient that, for some $u \in \widetilde{\text{Lip}}^1(R)$*

* This theorem was established in (2); see also (3).

with $\|u\|_{\sim} = 1$ the condition $u(x) - u(y) = r(x, y)$ be fulfilled if $\psi(e, e') \neq 0$, whatever the neighborhoods e, e' of the points x, y .

Proof. For any $\psi \in \Psi_\varphi$ and $u \in \widetilde{\text{Lip}}(R)$ with $\|u\|_\sim = 1$ (?)

$$\begin{aligned} \|\varphi\| &\geq \int_R u(x) \varphi(de) = \\ &= \iint_{RR} [u(x) - u(y)] \psi(de, de') \leq \iint_{RR} r(x, y) \psi(de, de') \geq \|\varphi\|. \end{aligned} \quad (4)$$

By Hahn's theorem, for some u equality holds in the first inequality; if ψ is an optimal displacement, then the last inequality is also replaced by equality, and then the double integrals are equal, and the condition of the theorem is fulfilled. Sufficiency also follows from (4).

Let us again suppose that φ_1 and φ_2 specify the existing and required distributions of masses in R , but $\varphi_1(R) < \varphi_2(R)$. Now, to pass from the existing state to the required one, it is necessary first to produce the missing mass $\varphi_2(R) - \varphi_1(R)$, whose distribution will be characterized by a nonnegative function $\varphi_3 \in \Phi(B)$, and then to perform a displacement taking the distribution $\varphi_1 + \varphi_3$ into φ_2 . The total costs of passing from the existing state to the required one are determined by the quantity

$$\int_R p(x) \varphi_3(de) + \iint_{RR} r(x, y) \psi(de, de'),$$

where $p(x)$ is the cost of producing a unit of mass at the point $x \in R$; $\psi \in \Psi_\varphi$; $\varphi = \varphi_1 + \varphi_3 - \varphi_2$. If the natural restrictions (3) are imposed on $p(x)$, then, using the general form of linear functionals in $\Phi(B)$, one can establish the following criterion.

Theorem 3. *For the optimality of a transition determined by the functions $\varphi_3 \in \Phi(B)$ and $\psi \in \Psi_\varphi$, it is necessary and sufficient that, for some function $u \in \text{Lip}^1(R)$ with $\|u\| = 1$, the conditions*

$$u(x) - u(y) = r(x, y), \quad u(z) = p(z)$$

be fulfilled, the first if $\psi(e, e') \neq 0$, whatever the neighborhoods e, e' of the points x, y ; the second if $\varphi_3(e'') \neq 0$ for any neighborhood e'' of the point z .

Let us now consider the problem of moving masses with restrictions on the cargo flows along individual routes.* Let φ_1 and φ_2 be the existing and required distributions of masses in R ; $\varphi_1(R) = \varphi_2(R)$; admissible displacements are characterized by functions $\psi \in \Psi_\varphi$ ($\varphi = \varphi_1 - \varphi_2$), for which

$$\psi(e, e') \leq \bar{\psi}(e, e') \quad (e, e' \in B),$$

where $\bar{\psi}(e, e')$ is a given positive function, completely additive in each argument, taking both finite and infinite values; here it is natural to assume that $\bar{\psi}(e, e') = \infty$ when $e \cap e' \neq \Lambda$.

Theorem 4. *For the optimality of an admissible displacement determined by the function ψ , it is necessary and sufficient that, for some function $u \in \widetilde{\text{Lip}}^1(R)$,*

the following conditions be fulfilled: 1) $u(x) - u(y) = r(x, y)$, if $0 < \psi(e, e') < \bar{\psi}(e, e')$, whatever the neighborhoods e, e' of the points x, y ; 2) $u(x) - u(y) \leq r(x, y)$, if $\psi(e, e') = 0$ for some neighborhoods e, e' of the points x, y ; 3) $u(x) - u(y) \geq r(x, y)$, if $\psi(e, e') = \bar{\psi}(e, e')$ for some neighborhoods e, e' of the points x, y .

4. The production-planning problem. There is a set $M = \{\mu\}$ of methods of production. The quantity of products produced by the μ -th method is characterized by a completely additive function φ_μ , defined on the system B of Borel sets of the metric compact space (of kinds) of products R (φ_μ takes both positive and negative values; the latter mean costs). Many—

* This problem for the case when R is a finite set was considered in (4).

the set $\{\varphi_\mu; \mu \in M\}$ is assumed to be bounded in $\Phi^V(B)$ and closed in $\Phi(B)$. Then the set M with the metric $r(\mu, \mu') = |\varphi_\mu - \varphi_{\mu'}|$ is compact. A production plan is characterized by a nonnegative completely additive function $h(e)$, defined on the Borel sets of the compact set M ; the quantity of products produced under this plan is determined by the function

$$\varphi^h = \int_M \varphi_\mu h(de).$$

We consider the problem* of constructing a plan satisfying the conditions $\varphi^h + \bar{\varphi}_1 \geq k\bar{\varphi}_2$, $k = \max$, where $\bar{\varphi}_1, \bar{\varphi}_2 \in \Phi(B)$ are given nonnegative functions, the first characterizing available resources and the second the required assortment. The desired plan is called **optimal**.

Theorem 5. *If the natural condition is fulfilled: $\varphi^h \geq 0$ entails $h(e) \equiv 0$ (meaning the absence of a nontrivial plan under which no costs are incurred), then there exists an optimal plan h_0 , and to it correspond such product valuations, determined by a function $u \in \text{Lip}^1(R)$ ($u(x) \geq 0$), that*

$$\int_R u(x)\varphi_\mu(de) \leq 0$$

for all $\mu \in M$, and for those μ for which $h_0(e) \neq 0$ for any neighborhood e of the point μ , equality holds in the last inequality. Conversely, if for some plan h_0 , $\varphi^{h_0} + \bar{\varphi}_1 \geq k\bar{\varphi}_2$, there correspond to it product valuations $u(x)$ having the indicated property, with $u(x) = 0$ if $\varphi^{h_0}(e) + \bar{\varphi}_1(e) > k\bar{\varphi}_2(e)$ for any neighborhood e of the point μ , then the plan h_0 is optimal.

Remark 1. A plan h is called **conditionally optimal** if $\varphi^h > -\bar{\varphi}_1$ and there is no plan h' for which $\varphi^{h'} + \bar{\varphi}_1 \geq k(\varphi^h + \bar{\varphi}_1)$ for some $k > 1$. Every conditionally optimal plan h is optimal for some function \bar{h}_2 ($\bar{h}_2 = \varphi^h + \bar{\varphi}_1$), and therefore to it there correspond product valuations of which the theorem speaks.

Remark 2. Applying Theorem 5 to the very special case when R consists of the collection of intervals of the real line

$$I_i = [2iT, (2i + 1)T] \quad (i = 1, 2, \dots, n),$$

we obtain the existence of an optimal plan and of the corresponding product valuations for every instant of time in the following problem.

There is a set $M = \{\mu\}$ of methods of producing products of n denominations, carried out during the time interval $[0, T]$. The quantity of products produced by the μ -th method is characterized by the function $\varphi_\mu \in \Phi(B)$ (if the interval $e = [\alpha, \beta] \in I_i$, then $\varphi_\mu(e)$ gives the quantity of the i -th product produced by the μ -th method during the time interval $[\alpha - 2iT, \beta - 2iT]$). The remaining conditions are the same as in the problem considered.

It follows from Remark 1 that to every conditionally optimal plan there also correspond product valuations for all instants of time.

Leningrad Branch
of the V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

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REFERENCES

1. V. I. Glivenko, *The Stieltjes Integral*, 1936.
2. L. V. Kantorovich, DAN, **37**, 227 (1942).
3. L. V. Kantorovich, *Uspekhi Mat. Nauk*, **3**, no. 2, 225 (1948).
4. L. V. Kantorovich, M. K. Gavurin, in: *Problems of Increasing the Efficiency of Transport Operation*, Publishing House of the Academy of Sciences of the USSR, 1949, pp. 110-138.
5. L. V. Kantorovich, DAN, **115**, no. 3 (1957).

* This problem, for the case when R and M are finite sets, is considered in (5).

Note: Figure translations are in progress. See original paper for figures.

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