UNIQUE DETERMINATION OF CERTAIN CONVEX SURFACES IN LOBACHEVSKY SPACE

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Abstract

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MATHEMATICS

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UNIQUE DETERMINATION OF CERTAIN CONVEX SURFACES IN LOBACHEVSKY SPACE

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Definition. A **cap** is a convex finite surface, homeomorphic to a plane, with planar boundary ( ), whose orthogonal projection onto the plane of the boundary does not go outside the domain bounded by the curve ( ).

Theorem 1. Let (F_1) be a convex cap of bounded specific curvature in Lobachevsky space; let (F_2) be a convex cap isometric to it.

Then (F_2) is either congruent to (F_1), or congruent to its mirror image.

Definition. By a **generalized horospherical cap in Lobachevsky space** we shall mean a convex finite surface (F), whose boundary curves (_1, _2, ..., _n) lie on horospheres (_1, _2, ..., _n), and such that, if from the points of each curve (_i) one draws rays orthogonal to the horosphere (_i) in the direction of its axes, then the surface consisting of the obtained half-cylinders (C_i) ((i = 1, 2, ..., n)) and the surface (F) is a complete convex surface.

In Lobachevsky space a finite convex domain on a convex surface may be homeomorphic to a sphere with (p) holes, where (p) is any nonnegative integer.

A. D. Aleksandrov proved that in Euclidean space every convex domain on a convex surface homeomorphic to a disk is isometric to a cap ((^4)).

We prove the theorems:

Theorem 2. Let (F) be a convex finite surface in Lobachevsky space; let (_1, _2, ..., _n) be rectifiable curves that are the boundaries of the surface (F), every arc of which, from the side of the surface (F), has rotation not less than (-s/k), where (k) is the radius of curvature of Lobachevsky space, and (s) is the length of the arc.

Then there exists a generalized horospherical cap (F’) isometric to the surface (F).

From Theorem 2 it follows:

Theorem 3. Every finite domain on a convex surface, convex in itself, is isometric to some generalized horospherical cap.

Next, a theorem on the unique determination of a generalized horospherical cap is proved.

Theorem 4. Let (F_1) be a generalized horospherical cap of bounded specific curvature; let (*{1,i}) ((i = 1, 2, ..., n)) be its boundary curves; let (F_2) be a generalized horospherical cap isometric to the surface (F_1); let (* {2,i}) be the boundary curve of (F_2) corresponding under the isometry to (*{1,i}). Suppose that the curves (* {1,k}), (*{2,k}), for (k = 2, 3, ..., n), are pairwise congruent or mirror-congruent smooth curves; (* {1,1}), (_{2,1}) are also smooth.

Then the surfaces (F_1) and (F_2) are congruent or mirror-congruent.

Definition. An **infinite convex surface with (p) infinitely distant points** is a complete infinite convex surface (F) in Lobachevsky space such that from any point of it one can draw only (p) distinct rays wholly belonging to the body bounded by the surface (F). The infinitely distant points determined by such rays will be regarded as belonging to (F).

Theorem 5. Let (F_1) be an infinite convex surface with a single infinitely distant point and of bounded specific curvature; let (F_2) be a convex surface isometric to (F_1). Suppose that on the surface (F_1) there is a sequence of closed curves (_i) whose lengths tend to zero, and such that every curve (*i) lies in a finite region of the surface (F_1) bounded by the curve (* {i+1}). Then the surfaces (F_1, F_2) can be superposed by a motion or by a motion with reflection.

Definition. An infinite convex surface (F) with (p) infinitely distant points, where (p) is any natural number, will be called an **acute convex infinite surface** if for each of its infinitely distant points (*∞) there exists a cylinder (Z) with closed directrix and generators passing through (* ∞), inside which the surface (F) is entirely contained.

Theorem 6. If an acute smooth convex infinite surface (F_1) of bounded specific curvature is isometric to a convex smooth surface (F_2), then the surface (F_2) is congruent or mirror-congruent to the surface (F_1).

The proof of the unique-determinacy theorems stated above rests essentially on the following lemma:

Lemma 1. Let (F_1) and (F_2) be two isometric oppositely oriented convex surfaces of bounded specific curvature in Lobachevsky space.

Then, if the locus of midpoints of the segments joining corresponding under the isometry points of (F_1) and (F_2) is a surface, it is a surface of generalized negative curvature.

This theorem for the case of surfaces in Euclidean space was proved by A. V. Pogorelov ((^1)). In generalizing the theorem to surfaces of Lobachevsky space, we follow A. V. Pogorelov at each separate stage of the proof, reducing the question to the case of Euclidean space. For this purpose we use each time the mapping of Lobachevsky space into the interior of a ball of Euclidean space according to Klein.

For the proof of Lemma 1 it was necessary to generalize a number of theorems to Lobachevsky space.

In particular, Theorems 7 and 8 were proved.

Theorem 7. The set of points ( ) of a convex surface (F) of Lobachevsky space which do not have Dupin’ s normal indicatrix has measure zero.

This theorem for convex surfaces of Euclidean space was proved by Busemann and Feller ((^2)).

Theorem 8. At every normal point (O) of a convex surface (F) of Lobachevsky space there exists the derivative of (k( )) with respect to any normal sequence (_m), and this derivative is equal to the product of the principal curvatures at the point (O), i.e.

$$\lim_{\sigma_m \to 0} \frac{k(\sigma_m)}{F(\sigma_m)} = k_1, k_2,$$

where (k( )) is the exterior curvature of the set ( ); (k_1, k_2) are the principal curvatures of the surface (F) at the point (O).

Theorem 7 for Euclidean space was proved by A. D. Aleksandrov (³). The proofs of these theorems reduce to the case of Euclidean space by means of the mapping mentioned above.

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CITED LITERATURE

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[3] A. D. Aleksandrov, Uchen. zap. Leningr. gos. univ., ser. matem., No. 37, issue 6, 1 (1939).
A. D. Aleksandrov, *The Intrinsic Geometry of Convex Surfaces*, Moscow-Leningrad, 1948.

Note: Figure translations are in progress. See original paper for figures.