

ON THE PROBLEM OF GENERAL ANAMORPHOSIS

The basic problem of nomography still remains the clarification of the conditions under which a given equation

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Abstract

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ON THE PROBLEM OF GENERAL ANAMORPHOSIS

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The basic problem of nomography still remains the clarification of the conditions under which a given equation

$$z = F(x, y) \tag{1}$$

can be represented in the form of a Soreau equation

$$\begin{vmatrix} \varphi_1(x) & f_1(x) & 1 \\ \varphi_2(y) & f_2(y) & 1 \\ \varphi_3(z) & f_3(z) & 1 \end{vmatrix} = 0, \tag{2}$$

which determines the scales of the nomogram from aligned points

$$\begin{aligned} u = \varphi_1(x), & \quad u = \varphi_2(y), & \quad u = \varphi_3(z), \\ v = f_1(x), & \quad v = f_2(y), & \quad v = f_3(z), \end{aligned} \tag{3}$$

which can be used for solving equation (1). Numerous works have been devoted to this problem (D' Ocagne, Barrow, Gronwall, Wilner, Smirnov, and many others), but there is still no effective method of solution. Moreover, it is natural to seek a path that automatically leads to an approximate nomogram in the case where an exact one is absent.

In the present paper we proceed from the assumption that $F(x, y)$, in the domain G under consideration of the x, y -plane, is differentiable a sufficient number of times and that $F_x F_y \neq 0$ in any subdomain \bar{G} of the domain G .

The method of solving the problem is as follows. We suppose that equation (1) is exactly nomographable. Having found the functions f_i, φ_i , we check whether the identity

$$\begin{vmatrix} \varphi_1(x) & f_1(x) & 1 \\ \varphi_2(y) & f_2(y) & 1 \\ \varphi_3\{F(x, y)\} & f_3\{F(x, y)\} & 1 \end{vmatrix} = 0. \quad (4)$$

is satisfied.

Equation (1) will sometimes be written in the form $x = \theta(y, z)$ or $y = \Phi(x, z)$.

Thus, if equation (1) is nomographable, then identity (4) holds. Differentiate it three times with respect to x . In the resulting equalities put $x = x_0$ and $f_1(x_0) = 0$, $f_1'(x_0) = 0$, $f_1''(x_0) = 2$, $f_1'''(x_0) = 0$, $\varphi_1(x_0) = 0$, $\varphi_1'(x_0) = 1$, $\varphi_1''(x_0) = 0$, $\varphi_1'''(x_0) = 0$. This can be done under the condition that, in the corresponding nomogram of equation (1), the scale x is curvilinear, which is characterized by the conditions

$$\begin{vmatrix} \bar{\varphi}_1'(x_0) & \bar{f}_1'(x_0) \\ \bar{\varphi}_1'''(x_0) & \bar{f}_1'''(x_0) \end{vmatrix} \neq 0, \quad \begin{vmatrix} \bar{\varphi}_1''(x_0) & \bar{f}_1''(x_0) \\ \bar{\varphi}_1'''(x_0) & \bar{f}_1'''(x_0) \end{vmatrix} \neq 0. \quad (5)$$

If the scale x is rectilinear, then one may put $\varphi_1^{(i)}(x_0) = 0$, $f_1(x_0) = 0$, $f_1'(x_0) = 1$, $f_1''(x_0) = 0$, $f_1'''(x_0) = a$.

Expanding determinant (4) and its derivatives, we obtain, for $x = x^0$:

$$\varphi_2(y)f_3\{F(x_0, y)\} - f_2(y)\varphi_3\{F(x_0, y)\} = 0; \quad (6)$$

$$\varphi_2(y)f_3^z F_x - f_2(y)(\varphi_3^z F_x - 1) - f_3 = 0; \quad (7)$$

$$\varphi_2(y)(f_3^{zz} F_x^2 + f_3^z F_{xx} - 2) - f_2(y)(\varphi_3^{zz} F_x^2 + \varphi_3^z F_{xx}) - 2(f_3^z F_x - \varphi_3) = 0, \quad (8)$$

$$\begin{aligned} \varphi_2(y)(f_3^{zzz} F_x^3 + 3f_3^{zz} F_x F_{xx} + f_3^z F_{xxx}) - f_2(y)(\varphi_3^{zzz} F_x^3 + 3\varphi_3^{zz} F_x F_{xx} + \varphi_3^z F_{xxx}) - \\ - 3(f_3^{zz} F_x^2 + f_3^z F_{xx} - 2\varphi_3^z F_x) = 0. \end{aligned} \quad (9)$$

Eliminating $\varphi_2(y)$ and $f_2(y)$, we arrive at the system (we omit the index 3)

$$\begin{aligned} \varphi'' f^2 - f'' \varphi f - 2f'(\varphi' f - \varphi f') + \frac{2}{F_x} [ff' + \varphi(\varphi' f - \varphi f')] + \\ + \frac{F_{xx}}{F_x^2} f(\varphi' f - \varphi f') = 0; \end{aligned} \quad (10)$$

$$\begin{aligned} \varphi''' f^2 - f''' \varphi f + 3\varphi'' f^2 \frac{F_{xx}}{F_x^2} - 3f'' \left[\varphi f \frac{F_{xx}}{F_x^2} - f \frac{1}{F_x} + \varphi' f - \varphi f' \right] + \\ + f(\varphi' f - \varphi f') \frac{F_{xxx}}{F_x^3} = 0. \end{aligned} \quad (11)$$

If here in F_x, F_{xx}, F_{xxx} one substitutes from equation (1) $y = \Phi(x_0, z)$, then this system may be regarded as a system of two ordinary differential equations with two unknown functions $f_3(z)$ and $\varphi_3(z)$. It is reducible to a system of two equations of the 2nd order, since, differentiating (10) with respect to z and subtracting from (11), we obtain

$$\begin{aligned} -\varphi'' \left(\frac{2}{F_x} \varphi f - 2 \frac{F_{xx}}{F_x^2} f^2 \right) + \left(2\varphi f \frac{F_{xx}}{F_x^2} - \frac{f}{F_x} - \frac{2}{F_x} \varphi^2 \right) f'' + \\ + \varphi' \left(\frac{F_{xx}}{F_x^2} f^2 + \frac{2}{F_x} \varphi f \right)' + f' \left(\frac{2}{F_x} f - \frac{2}{F_x} \varphi^2 - \frac{F_{xx}}{F_x^2} \varphi f \right)' - \\ - (\varphi' f - \varphi f') f \frac{F_{xxx}}{F_x^3} = 0. \end{aligned} \quad (12)$$

In the general case, from this system the functions $f_3(z)$ and $\varphi_3(z)$ are determined, depending on 4 constants of integration. Exceptional cases of non-uniqueness of the solution of the system of differential equations have not been considered by us.

From equations (6) and (7) we determine $\varphi_2(y)$ and $f_2(y)$ in terms of the same 4 constants. Requiring that equality (4) be satisfied for $y = y_0$ and $y = y_1$, we find the functions $\varphi_1(x)$ and $f_1(x)$. The indicated constants will also enter into them. Substituting the functions found into equality (4), we obtain an identity for determining the constants.

If, by the method presented, one considers the possibility of representing equation (1) in the form

$$f_1(x) + f_2(y) = f_3(z), \quad (13)$$

then we obtain

$$f_3(z) = \int \frac{dz}{F_x \{x_0, \Phi(x_0, z)\}}; \quad (14)$$

$$f_1(x) = f_3\{F(x, y_0)\}; \quad (15)$$

$$f_2(y) = f_3\{F(x_0, y)\} - f_1(x_0). \quad (16)$$

To represent equation (1) by an equation of Cauchy type with a curvilinear scale y : $f_1(x) + f_2(y)f_3(z) = \varphi_2(y)$, we obtain:

$$f_1(x) = \exp \left[a \int \frac{F_x(x, y_0)}{F_y(x, y_0)} dx \right]; \quad (17)$$

$$f_3(z) = f_1\{\theta(y_0, z)\}. \quad (18)$$

To determine the functions $\varphi_2(y)$ and $f(y)$ we have the identities

$$f_1(x_1) + f_2(y)f_3\{F(x_1, y)\} = \varphi_2(y); \quad (19)$$

$$f_1(x_2) + f_2(y)f_3\{F(x_2, y)\} = \varphi_2(y), \quad (20)$$

where x_1 and x_2 are arbitrary values of $x \in \overline{G}$.

Finally, to determine the constant a we use the identity obtained by eliminating $\varphi_2(y)$ and $f_2(y)$:

$$\begin{vmatrix} f_1(x_1) & f_3(z_1) & 1 \\ f_1(x_2) & f_3(z_2) & 1 \\ f_1(x) & f_3(z) & 1 \end{vmatrix} = 0, \quad (21)$$

where the three pairs of numbers x_i, z_i satisfy equation (1) for one value of y , different from y_0 .

To represent equation (1) by the Clark equation

$$f_1 f_2 f_3 + \varphi_2(f_1 + f_3) + \psi_2 = 0, \quad (22)$$

we obtain, for finding the function $f_1(x)$, the equality

$$\int \frac{df_1}{f_1^2 a + f_1 b + c} = - \int \frac{F_x(x, y_0)}{F_y(x, y_0)} dx.$$

Thus, in this case as well the matter reduces to determining constants from algebraic equations.

The equation of the rectilinear scale for the case of an equation of the 5th nomographic order is determined from the differential equation

$$f_3'' f_3 = 2f_3'^2 - f_3 f_3' \frac{F_{xx}\{x_0, \Phi(x_0, z)\}}{F_x^2\{x_0, \Phi(x_0, z)\}} - \frac{2}{F_x\{x_0, \Phi(x_0, z)\}} f_3'.$$

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Note: Figure translations are in progress. See original paper for figures.

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