MATHEMATICS

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Abstract

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ON ELLIPTIC EQUATIONS CONTAINING SMALL PARAMETERS IN THE HIGHEST DERIVATIVES

1. Partial differential equations containing a small parameter have attracted the attention of many investigators ($^{1-6}$).

In the linear case, for such equations one poses, for example, the following problem:

A family of operators depending on the parameter ε, defined in a domain Q of the space $(x_1, \dots, x_n)$, is given:

$$L_{r,\varepsilon} u = \sum_{s=p}^{r} \varepsilon^{s-p} L_{su},$$

where L_{su} is a differential operator of order $\leq s$, which for simplicity we shall assume not to depend on ε.

Under the corresponding boundary conditions, one considers solutions $u_\varepsilon(x)$ of the equations

$$L_{r,\varepsilon} u = h, \tag{1}$$

and studies the asymptotics of the solutions $u_\varepsilon(x)$ for small ε and their relation to some solution $w(x)$ of the equation

$$L_{pw} = h. \tag{2}$$

In the present note we consider the case when r and p are even numbers: $p = 2k,\ r = 2(k + l)$, and the operators $L_{r,\varepsilon}$ and L_p are elliptic. The solution $u_\varepsilon(x)$ of equation (1) is considered in the domain Q with boundary q, on which homogeneous boundary conditions of the first boundary-value problem are prescribed (nonhomogeneous boundary conditions cause no difficulties):

$$u\Big|_q = \dots = \frac{\partial^{k-1}u}{\partial n^{k-1}}\Big|_q = 0, \tag{3}$$

$$\frac{\partial^k u}{\partial n^k}\Big|_q = \dots = \frac{\partial^{k+l-1}u}{\partial n^{k+l-1}}\Big|_q = 0. \tag{4}$$

By $w(x)$ we shall denote the solution of equation (2) under conditions (3).

We impose the following conditions on the operators $L_{r,\varepsilon}$ and L_p:

I. $L_p(u)$ $(p = 2k)$, under conditions (3), has a positive quadratic form:

$$(L_{pu}, u) \geq c^2 \sum (D^{ku}, D^{ku}),$$

where D^k is a partial differential operator of order k. (Instead of this condition one could merely require that problem (2), (3) be solvable, i.e. that zero not be an eigenvalue of the operator L_p.)

II. Let $\pi_{2j}(\xi, x)$, $\xi = (\xi_1, \dots, \xi_n)$, be the characteristic form of the operator L_{2j} (corresponding, evidently, only to the terms of order $2j$ of this operator), and

$$\pi_\varepsilon(\xi, x) = \sum_{j=k+1}^{k+l} \varepsilon^{2j} \pi_{2j}(\xi, x).$$

We require that

$$\pi_\varepsilon(\xi, x) \geq c_1^2 \left(\varepsilon^{2(k+l)} |\xi|^{2(k+l)} + \varepsilon^{2(k+1)} |\xi|^{2(k+1)}\right). \tag{5}$$

We note that from conditions I and II there follow the ellipticity of the operators $L_{r,\varepsilon}$ and L_{2k} and the solvability of problem (1), (3), (4) for small ε.

As is shown below, the difference $v_\varepsilon(x)$ between $u_\varepsilon(x)$ and $w(x)$ has the character of a boundary layer of order k. One of the aims of this note is the constructive and, as far as possible, elementary construction of the boundary layer $v_\varepsilon(x)$, which can be extended also to other problems with a small parameter. For the problem considered in the present note, this construction of v_ε reduces to the solution of an ordinary differential equation with constant coefficients.

2. For simplicity of exposition we shall restrict ourselves to the case where the coefficients and the right-hand side of equation (2), as well as the boundary q, are infinitely differentiable. From the course of the exposition one can derive sufficient conditions, in the sense of the order of differentiability of the coefficients and the boundary, under which the results presented are valid.

In investigating the asymptotics of u_ε it is useful to introduce, in a neighborhood of the boundary, a local coordinate system $(\rho, \varphi_1, \dots, \varphi_{n-1})$, where $\rho = 0$ is the equation of the boundary q, $\varphi = (\varphi_1, \dots, \varphi_{n-1})$ determines a point on q, and the system of coordinate lines ρ forms with q an angle different from zero. For example, as the coordinate lines ρ one may take the normals to q. We shall construct the boundary layer in the strip adjacent to q, $Q_\delta :\ 0 \le \rho \le \delta$, where δ is some positive constant. The operator L_s in the new coordinate system has the form: $L_{su} = a_s(\rho, \varphi)\, \partial^s u / \partial \rho^s + \cdots$, and here

$$a_s(\rho, \varphi) = a_s(\varphi) + \rho a_{s,1}(\varphi) + \cdots + \rho^N a_{s,N}(\varphi) + \varepsilon^{N+1} a_{s,N+1}(\rho, \varphi); \tag{6}$$

the remaining coefficients of the operator L_{su} are expanded analogously.

Introduce the new variable $t = \rho/\varepsilon$; then $\dfrac{\partial^p}{\partial \rho^p} = \dfrac{1}{\varepsilon^p}\dfrac{\partial^p}{\partial t^p}$. Therefore

$$\varepsilon^{2k} L_{r,\varepsilon} u = M_0 u + \sum_{p=1}^{N} \varepsilon^p R_{pu} + \varepsilon^{N+1} R_{N+1} u, \tag{7}$$

where

$$M_0 u = \sum_{j=2k}^{2(k+l)} a_j(\varphi) \frac{\partial^j u}{\partial t^j}; \tag{8}$$

R_{pu} are linear differential operators with bounded coefficients, for $p \le N$ depending polynomially on $t = \rho/\varepsilon$. Consider the equation $M_0 u = 0$ and its characteristic equation $P_\varphi(\lambda) \equiv \sum_{j=2k}^{2(k+l)} a_j(\varphi) \lambda^j$.

From conditions I and II it follows

Lemma 1. *The equation $P_\varphi(\lambda) = 0$ has l roots with negative real part, $-\lambda_i(\varphi)$, $i = 1, 2, \dots, l$ (for simplicity we assume them distinct), and $2k$ roots equal to 0.*

3. **Construction of v_ε.** We shall successively construct, in a neighborhood of q, functions $v_0, v_1, \dots, v_N$, where v_0 is the solution of the equation

$$M_0 v_0 \equiv \sum_{j=2k}^{2(k+l)} a_j(\varphi) \frac{\partial^j v_0}{\partial t^j} = 0 \tag{9}$$

under the boundary conditions

$$\left.\frac{\partial^{k+\tau} v_0}{\partial t^{k+\tau}}\right|_{t=0} \equiv \varepsilon^{k+\tau} \left.\frac{\partial^{k+\tau} v_0}{\partial \rho^{k+\tau}}\right|_{\rho=0} = -\, \varepsilon^{k+\tau} \left.\frac{\partial^{k+\tau} w}{\partial \rho^{k+\tau}}\right|_{\rho=0} \quad (\tau = 0, 1, \dots, l-1). \tag{10}$$

We seek v_0 as a function of boundary-layer type (i.e. noticeably different

from zero only near the boundary of the domain), namely, in the form of the linear combination

$$v_0 = \varepsilon^k \sum_{j=1}^{l} c_j(\varphi) e^{-\lambda_j(\varphi)t} = \varepsilon^k \sum_{j=1}^{l} c_j(\varphi) e^{-\lambda_j(\varphi)\rho/\varepsilon}. \tag{11}$$

To satisfy conditions (10), one must solve for $c_j(\varphi)$ a system of linear equations with determinant different from zero. The function v_0 satisfies conditions (3) up to a quantity of order ε.

If $v_0, \dots, v_{i-1}$ are known, then v_i is determined from the inhomogeneous equation

$$M_0 v_i = -\sum_{j=1}^{i} R_j(v_{i-j}) \tag{12}$$

under the boundary conditions

$$\partial^{k+\tau} v_i / \partial t^{k+\tau}\big|_{t=0} = 0 \qquad (\tau = 0, \dots, l-1)$$

and from the requirement that v_i be a boundary-layer type function. Elementary inductive arguments show that

$$v_i = \varepsilon^k \sum_{j=1}^{l} d_j(t, \varphi) e^{-\lambda_j(\varphi)t} \equiv \varepsilon^k \sum_{j=1}^{l} d_j\left(\frac{\rho}{\varepsilon}, \varphi\right) e^{-\lambda_j(\varphi)\rho/\varepsilon}, \tag{13}$$

where d_j depends polynomially on $t = \rho/\varepsilon$.

Denoting $v_\varepsilon = v_0 + \varepsilon v_1 + \dots + \varepsilon^N v_N$, we obtain, by virtue of (7), (9), and (12):

$$L_{r,\varepsilon} v_\varepsilon = \varepsilon^{N+1-2k}\{R_{N+1} v_0 + [R_N v_1 + \varepsilon R_{N+1} v_1] + \cdots$$

$$\cdots + [R_1 v_N + \varepsilon R_2 v_N + \dots + \varepsilon^N R_{N+1} v_N]\}.$$

Putting $N = k$, we have in the boundary strip: $L_{r,\varepsilon} v_\varepsilon = O(\varepsilon)$; $v_\varepsilon + w$ satisfies conditions (4), and conditions (3) up to ε. One may add to v_ε the function $\varepsilon\alpha(\rho, \varphi)$—a polynomial of degree $k-1$ with respect to $t = \rho/\varepsilon$, such that $v_\varepsilon + \varepsilon\alpha(\rho, \varphi)$ satisfies conditions (3) and $L_{r,\varepsilon}(v_\varepsilon + \varepsilon\alpha) = O(\varepsilon)$.

Denoting by $\psi(\rho)$ a function with bounded derivatives of arbitrary order, equal to 1 for $0 \le \rho \le \frac{1}{2}\delta$ and equal to 0 for $\rho > \delta$, and by v_ε^* and α^* the functions equal, respectively, to ψv and $\psi\alpha$, we have:

$$L_{r,\varepsilon}[w+v_\varepsilon^*+\varepsilon\alpha^*]=h+O(\varepsilon),$$

where $w+v_\varepsilon^*+\varepsilon\alpha^*$ satisfy the boundary conditions (3), (4).

Repeating the calculations indicated above, one can obtain a more accurate asymptotic expansion (taking $w=w_0,\ \alpha^*=\alpha_0^*$):

$$u_\varepsilon=(w_0+v_\varepsilon^*+\varepsilon\alpha_0^*)+\varepsilon(w_1+v_{1\varepsilon}^*+\varepsilon\alpha_1^*)+\cdots$$

$$\cdots+\varepsilon^s(w_s+v_{s\varepsilon}^*+\varepsilon\alpha_s^*)+\beta_s, \tag{14}$$

where β_s satisfies the equation

$$L_{r,\varepsilon}(\beta_s)=O(\varepsilon^{s+1}), \tag{15}$$

and the function β_s satisfies conditions (3), (4). Here the i-th derivatives of the right-hand side of (15) have order $O(\varepsilon^{s+1-i})$. From the estimates proved it follows that β_s and its derivatives up to order $2k$ have, in the L_2 norm, order ε^{s+1}, while each subsequent differentiation does not change the order (ε^{s+1}) in any closed subdomain and decreases the order

by ε (in the sense of the norm in L_2) to unity in all of $\overline{Q}$. By virtue of the embedding theorems of S. L. Sobolev [7], estimates of arbitrary order in ε can be obtained in the C-metric.

Theorem. The solution u_ε of problem (1), (3), (4) can be represented by formula (14), where w_0 is the solution of problem (2), (3), w_i is the solution of problem (2), (3) with the right-hand side h replaced by

$$-\sum_{j=1}^{[i]}L_{2k+j}(w_{i-1})-\sum_{j=1}^{[i]+1}L_{2k+j-1}(\alpha_j^*),$$

$[i]=\min(i,2l)$, v_j^* are functions of boundary-layer type; α_i^* are bounded functions. The remainder β_s has the smallness of the order indicated above.

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Note: Figure translations are in progress. See original paper for figures.