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Abstract

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ON THE STABILITY OF PERIODIC SOLUTIONS OF QUASILINEAR AUTONOMOUS SYSTEMS WITH MANY DEGREES OF FREEDOM

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In the present paper we consider the question of the stability of periodic solutions of a quasilinear autonomous system with many degrees of freedom under the assumption that, among the roots λ_s of the characteristic equation corresponding to the linear system, there is a multiple zero root, and also an arbitrary number of purely imaginary roots of any multiplicity. The real parts of all the remaining roots are assumed to be positive, and the elementary divisors corresponding to all multiple roots are linear. One of the dynamical systems whose motion is described by equations of this type was considered by us in ([1]). Here some results of that paper are generalized. Special cases of the problem were studied in ([2,3]); the case of a non-autonomous system was considered in ([4]).

The system under consideration can be reduced to the canonical form

$$\frac{dx_s}{dt} = \lambda_s x_s + \mu F_s(x_1, \dots, x_l, \mu) \quad (s = 1, \dots, l), \tag{1}$$

where F_s are analytic functions of $x_1, \dots, x_l$ in a domain G, inside which lie the solutions studied below, and of the parameter $\mu \geq 0$ for sufficiently small values of it. One may assume that to complex-conjugate λ_s there correspond complex-conjugate F_s and solutions x_s.

As is known ([5,6]), periodic solutions of system (1) with period $T_{q,r}$, close to $T^0_{q,r} = 2\pi q/\nu_r$ ($q \neq 0$ is an integer), may correspond to any pair of purely imaginary roots $\lambda_{r-1} = \bar{\lambda}_r = i\nu_r$ of the characteristic equation. In considering the question of existence and stability of solutions with a period close to any of the indicated periods, one can always, by a change of the independent variable, reduce the problem to the study of a periodic solution with period close to 2π. Assuming that this has been done, let us divide all the purely imaginary roots λ_s into groups, each of which combines, first, all multiple roots and, second, all roots differing by qi.

Denote by $\lambda_s^{(j)} = i\nu_s^{(j)}$ a purely imaginary root from group j. We single out into a special group the critical roots, i.e. roots of the form $\lambda_s = in_s$, where n_s are integers or zero, at least one pair of which is nonzero, since otherwise the period of the solution under consideration would be undetermined. The imaginary parts ν_s of the roots of all the remaining groups differ from integers and from zero. Thus we have:

$$\lambda_s = \begin{cases} in_s, & s = 1, \dots, m; \\ i\nu_s^{(j)}, & s = m + m_1 + \dots + m_{j-1} + 1, \dots, m + m_1 + \dots + m_j \\ & (j = 1, \dots, p); \\ -u_s + i\nu_s, & u_s > 0, \quad s = m + m_1 + \dots + m_p + 1, \dots, l. \end{cases} \tag{2}$$

Here p is the number of groups of purely imaginary roots; m_j is the number of roots in the j-th group, and, in accordance with the principle of dividing the roots into groups,

$$\left|\gamma_r^{(j)} - \gamma_s^{(j)}\right| = n; \qquad \left|\gamma_r^{(j)} - \gamma_s^{(\sigma)}\right| \neq n, \quad j \neq \sigma, \tag{3}$$

where n is any integer or zero.

For $\mu = 0$, equations (1) admit a solution with period 2π

$$x_s^0 = \begin{cases} \alpha_s e^{in_s t}, & s = 1, \dots, m; \\ 0, & s = m + 1, \dots, l, \end{cases} \tag{4}$$

depending on m arbitrary parameters α_s. If there exists a periodic solution of system (1) which for $\mu = 0$ turns into (4), then the period of this solution T for $\mu \neq 0$ will, generally speaking, be different from 2π. Denoting $T = T(\mu) = 2\pi(1 - \mu\delta_1)$, where $\delta_1 = \delta_1(\mu)$ is an unknown function of μ, and making the change of independent variable by the formula $t = \tau(1 - \mu\delta_1)$, we bring system (1) to the form

$$\dot{x}_s = (1 - \mu\delta_1)\left[\lambda_s x_s + \mu F_s(x_1, \dots, x_l, \mu)\right], \quad s = 1, \dots, l, \tag{5}$$

where the dot denotes differentiation with respect to τ. Thus, the investigation of the periodic solutions of system (1) with period $T(\mu)$ is reduced to the investigation of the solutions of system (5) with period 2π.

One may always assume that at least one of the products $\alpha_s\lambda_s$ $(1 \leq s \leq m)$ is nonzero, since otherwise the period of the solution (4) would be indeterminate. Let, specifically, $\alpha_m\lambda_m \neq 0$. Then, slightly modifying the results of paper (5), it can be shown that in the given case the periodic solutions of period 2π of system (5) can correspond only to those values $\alpha_1, \dots, \alpha_m$ which satisfy the equations

$$Q_s = \alpha_m \lambda_m P_s - \alpha_s \lambda_s P_m = 0, \quad s = 1, \dots, m-1, \tag{6}$$

where

$$P_s = P_s(\alpha_1, \dots, \alpha_m) = \int_0^{2\pi} F_s(\alpha_1 e^{\lambda_1 \tau}, \dots, \alpha_m e^{\lambda_m \tau}, 0, \dots, 0) e^{-\lambda_s \tau} \, d\tau. \tag{7}$$

Since the number of equations (6) exceeds the number of unknowns by one, one of the unknowns, for example α_m, may be prescribed arbitrarily. In particular, if it is stipulated that $\lambda_m = \bar{\lambda}_{m-1}$, then it is convenient to take $\alpha_m = \bar{\alpha}_{m-1} = \alpha$.

To each system of values of the $m-1$ constants $\alpha_1^*, \dots, \alpha_{m-2}^*, \alpha_{m-1}^* = \alpha_m^* = \alpha^*$, satisfying (6), there indeed corresponds one unique solution of equations (1), periodic and analytic for sufficiently small μ, if

$$\left| \frac{\partial(\tilde{Q}_1, \dots, \tilde{Q}_{m-1})}{\partial(\alpha_1, \dots, \alpha_{m-2}, \alpha)} \right| \Bigg|_{\alpha_s = \alpha_s^*} \neq 0, \quad \tilde{Q}_s = Q_s(\alpha_1, \dots, \alpha_{m-2}, \alpha, \alpha). \tag{8}$$

In this case the correction to the frequency $\delta_1(0) = \delta^*$ will be

$$\delta^* = P_m / 2\pi \alpha_m^* \lambda_m. \tag{9}$$

Turning to the investigation of the stability of periodic solutions, we shall prove the following theorem.

Theorem. If, for a certain system of values of the constants $\alpha_1 = \alpha_1^*, \dots, \alpha_{m-2} = \alpha_{m-2}^*, \alpha_{m-1} = \alpha_m = \alpha^*$, satisfying equations (6), the real parts of all roots χ of the algebraic equations

$$D_0(\chi) = \left| \frac{\partial Q_s}{\partial \alpha_\sigma} - \delta_{s\sigma} \lambda_m \chi \right| = 0, \qquad s, \sigma = 1, \dots, m-1; \tag{10}$$

$$D_j(\chi) = \left| \int_0^{2\pi} \left(\frac{\partial F_s}{\partial x_\sigma} \right) e^{(\nu_\sigma^{(j)} - \nu_s^{(j)}) i \tau} \, d\tau - \delta_{s\sigma} (2\pi i \nu_s^{(j)} \delta^* + \chi) \right| = 0, \tag{11}$$

$$s, \sigma = m + m_1 + \dots + m_{j-1} + 1, \dots, m + m_1 + \dots + m_j \quad (j = 1, \dots, p)$$

$$\left(\delta_{s\sigma} \text{ is the Kronecker symbol,} \qquad \left(\frac{\partial F_s}{\partial x_\sigma} \right) = \frac{\partial F_s}{\partial x_\sigma} \Bigg|_{x_r = x_r^0, \ \mu = 0} \right)$$

are negative, then, for sufficiently small μ, this system of constants corresponds to one single periodic solution of equations (1), analytic with respect to μ and asymptotically stable, which for $\mu = 0$ turns into the solution (4). If the real part of at least one of the roots of equations (10) or (11) is positive, then the corresponding solution is unstable; in the presence of purely imaginary or zero roots, an additional investigation is necessary.

Proof. The characteristic equation of the system in variations corresponding to equations (5) can be represented in the form

$$D = \left|\delta_{s\sigma}(e^{2\pi\lambda_s} - \rho) + \mu e^{2\pi\lambda_s}(C_{s\sigma} - 2\pi\lambda_s\delta^*\delta_{s\sigma}) + \cdots\right| = 0, \tag{12}$$

$$s, \sigma = 1, \dots, l,$$

where

$$C_{s\sigma} = \int_0^{2\pi} \left(\frac{\partial F_s}{\partial x_\sigma}\right) e^{(\lambda_\sigma - \lambda_s)\tau}\, d\tau, \qquad s, \sigma = 1, \dots, l, \tag{13}$$

and, according to equalities (2), (4), and (7), in particular we have:

$$C_{s\sigma} = \left.\frac{\partial P_s}{\partial \alpha_\sigma}\right|_{\alpha_r = \alpha_r^*}, \qquad s, \sigma = 1, \dots, m. \tag{14}$$

For $\mu = 0$, all off-diagonal elements of the determinant D vanish, and equation (12) has the roots $\rho_s = \rho_s^0 = e^{2\pi\lambda_s}$. The moduli of the first $m + m_1 + \cdots + m_p$ roots, according to (2), are equal to unity, while those of all the others are less than unity. For sufficiently small μ, the moduli of these latter roots remain less than unity, and the question of stability reduces to the study of the following approximation to the first $m + m_1 + \cdots + m_p$ roots.

Let us first consider the group of m critical roots, for which $\rho_s^0 = e^{2\pi i n_s} = 1$. The arguments and calculations analogous to those carried out in (4) show that these roots can be sought in the form $\rho = 1 + \mu\chi + \mu R(\mu)$, where $R(0) = 0$. To determine the coefficients χ, we obtain the algebraic equation of degree m

$$|C_{s\sigma} - (2\pi\lambda_s\delta^* + \chi)\delta_{s\sigma}| = 0, \qquad s, \sigma = 1, \dots, m. \tag{15}$$

Denote by $P_s^*(h)$ the result of substituting in (7) $e^{\lambda_\sigma(\tau+h)}$ instead of $e^{\lambda_\sigma\tau}$. Then we shall have: $P_s^*(h) \equiv e^{\lambda_s h} P_s^*(0)$. Differentiating this identity with respect to h and then putting $h = 0$, we obtain, taking into account (6) and (9), the relations

$$\left(\sum_{\sigma=1}^{m}\frac{\partial P_s}{\partial\alpha_\sigma}\alpha_\sigma\lambda_\sigma-\lambda_s P_s\right)_{\alpha_r=\alpha_r^*}=\sum_{\sigma=1}^{m}C_{s\sigma}\alpha_\sigma^*\lambda_\sigma-2\pi\lambda_s^2\alpha_s^*\delta^*=0, \tag{16}$$

from which it follows that one of the roots of equation (15) is always equal to zero. This was to be expected, since, owing to the autonomous character of system (1), equation (12) must have the root $\rho_1 = 1$; it is known[7] that the presence of one such root does not affect the question of stability.

Separating off the zero root of equation (15), we arrive at equation (10), from which all the remaining roots χ are found.

It is proved analogously that the other roots of (12), for which $|\rho_s^0| = 1$, may be sought in the form

$$\rho = e^{2\pi i\nu_s^{(j)}}[1+\mu\chi^{(j)}+\mu R(\mu)],$$

where $R^{(j)}(0) = 0$, and the first approximations $\chi^{(j)}$ to the roots of the j-th group ($j = 1, \dots, p$) are determined from the algebraic equations of degree m_j, (11). But, for sufficiently small μ, the conditions of asymptotic stability $|\rho_s| < 1$, $s = 2, \dots, m+m_1+\dots+m_p$, are equivalent to the inequalities $\operatorname{Re}\chi < 0$, and the conditions of instability $|\rho_s| > 1$ to the inequalities $\operatorname{Re}\chi > 0$; when $\operatorname{Re}\chi = 0$, to decide the question of stability it is necessary to know $|\rho|$ with a higher degree of accuracy.

To complete the proof it remains to note that the inequality to zero of the real parts of the roots of equation (10) ensures the fulfillment of condition (8). This follows from the fact that, owing to the relations

$$\sum_{\sigma=1}^{m}\left(\frac{\partial Q_s}{\partial\alpha_\sigma}\alpha_\sigma\lambda_\sigma\right)_{\alpha_r=x_r^*}=0 \quad (s=1,\dots,m-1),$$

which follow from (8), (9), (14), and (16), the determinant (8) differs from $D_0(\chi)$ only by a nonzero factor.

Thus, as in the case of a nonautonomous system[4], the investigation of stability is reduced to the Hurwitz problem for algebraic equations whose coefficients are expressed directly through the right-hand sides of the first $m+m_1+\dots+m_p$ equations (1) and require for their calculation only knowledge of the generating solution.

In addition to the theorem proved, one can show that the number of equations (11) that actually have to be considered is decreased by no less than $\frac{1}{2}(p-1)$ units; this follows from the assumption that if $\lambda_{r-1} = \bar{\lambda}_r$, then $F_{r-1} = \bar{F}_r$ and $\chi_{r-1} = \bar{\chi}_r$.

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Note: Figure translations are in progress. See original paper for figures.