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Abstract

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ON AN EQUATION OF ELLIPTIC TYPE DEGENERATING ON THE BOUNDARY OF THE DOMAIN

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Consider the equation

$$L(u) \equiv Ju_{yy} + u_{xx} + au_y + bu_x + cu = 0, \tag{1}$$

where $a(x,y)$, $b(x,y)$, $c(x,y)$ are analytic functions of the independent variables x and y in any finite part of the half-plane $y \geq 0$, and $c(x,y) \leq 0$.

In the half-plane $y > 0$, equation (1) is of elliptic type; it degenerates for $y = 0$.

Let D be the domain bounded by the segment AB of the axis Ox and by a smooth open arc Γ issuing from the points A and B and lying in the half-plane $y > 0$.

It is known ([1]) that, depending on whether $a(x,0) < 1$ or $a(x,0) \geq 1$ on AB, in the class of bounded functions there are uniquely solvable, respectively, either the Dirichlet problem—the determination of a solution of equation (1) from continuous data on $\Gamma + AB$—or the so-called problem E—the determination of a solution of equation (1) from continuous data on Γ.

Below we give, in a certain sense, a constructive characterization of solutions of equation (1) near the boundary of degeneration of type and attempt to investigate boundary-value problems in the formulation proposed by A. V. Bitsadze ([2]).

1. Consider the function

$$\omega(x,y) = \int_y^1 \exp\left[\int_t^1 a(x,r)r^{-1}\,dr\right] dt + C_0.$$

We choose the constant C_0 so that $\omega > 0$ in the closed domain $\overline{D}$. The function $\omega(x,y)$ is an analytic function for finite values of x and $y > 0$.

By integration by parts it is easy to show that for $y < 1$

$$\omega(x,y) = \psi(x,y) \exp\left[\int_y^1 a_1(x,r)\,dr\right](1+g(x,y)), \tag{2}$$

where

$$a_1(x,y) = \frac{a(x,y)-a_0(x)}{y}, \qquad a_0(x) = a(x,0);$$

$$\psi(x,y) = \begin{cases} \dfrac{y^{1-a_0(x)}-1}{a_0(x)-1}, & \text{if } a_0(x) \neq 1, \\ -\log y, & \text{if } a_0(x) = 1. \end{cases}$$

$$g(x,y) = \frac{C_0}{\psi(x,y)} \exp\left[-\int_y^1 a_1(x,r)\,dr\right] -$$

$$-\frac{1}{\psi(x,y)} \int_y^1 \psi(x,t) a_1(x,t) \exp\left[-\int_y^t a_1(x,r)\,dr\right] dt > -1,$$

moreover

$$g(x,y) = \frac{1}{\psi(x,y)} \int_y^1 \psi(x,t)\,dt \cdot O(1). \tag{3}$$

From (2) and (3) it is clear that $\omega(x,y)$ has singularities at those points of the segment AB where $a_0(x) \geqslant 1$.

Theorem 1. *For any continuous function $f(x,y)$ prescribed on $\Gamma + AB$, there exists a unique twice continuously differentiable solution $u(x,y)$ of equation (1), satisfying the condition*

$$\lim_{(x,y)\to Q} \frac{u(x,y)}{\omega(x,y)} = f, \qquad Q \in \Gamma + AB.$$

For the proof, first consider the function

$$W(x,y) = \omega(x,y)\left(1 + \left(\log\frac{R}{y}\right)^{-\alpha}\right) + N - (x+\delta)^m,$$

where $0 < \alpha < 1$; N and m are positive numbers; $x + \delta > 1$ and $R > y$ in $\overline{D}$. By direct calculation one can show that

$$L(W)=\alpha\omega y^{-1}\left(\log\frac{R}{y}\right)^{-\alpha-2}\left[\alpha+1+\left(a_0(x)-1+2y\frac{\omega_y}{\omega}\right)\log\frac{R}{y}+o(\sqrt{y})\right]-$$

$$-m(x+\delta)^{m-2}[m-1+b(x+\delta)]+cW. \tag{4}$$

To obtain equality (4) one must use:

$$\frac{\omega_x}{\omega}=O(|\log y|),\qquad \frac{\omega_{xx}}{\omega}=O(|\log y|^2),\qquad y\omega_{yy}+a\omega_y=0. \tag{5}$$

On the basis of (2) and (3), for $y<1$ we have

$$a_2(x,y)=a_0(x)-1+2y\frac{\omega_y}{\omega}=$$

$$=a_0(x)-1-2\frac{y^{1-a_0(x)}}{\psi(x,y)}+\frac{y^{1-a_0(x)}}{\psi^2(x,y)}\int_y^1\psi(x,t)\,dt\cdot O(1). \tag{6}$$

It is easy to show that, for small values of y,

$$a_0(x)-1-2\frac{y^{1-a_0(x)}}{\psi(x,y)}\leqslant\frac{2}{\log y}. \tag{7}$$

In view of (6) and (7), for small y we shall have

$$\alpha+1+a_2(x,y)\log\frac{R}{y}+o(\sqrt{y})<0. \tag{8}$$

On the basis of (8) and (4), m and N can be chosen so large that $W>0$ and $L(W)<0$ in D. We now prove uniqueness. Let $u(x,y)$ be a solution of equation (1) satisfying the condition

$$\lim_{(x,y)\to Q}\frac{u(x,y)}{\omega(x,y)}=0,\qquad Q\in\Gamma+AB.$$

We note that if some function $v(x,y)$ satisfies in D the condition $L(v)<0$, then it cannot have a negative minimum inside D.

Consider the function $\varepsilon W+u$, where $\varepsilon>0$ is an arbitrary number. It is easy to see that for any $\varepsilon>0$, $L(\varepsilon W+u)<0$ in D.

Since $\varepsilon W+u>0$ on $\Gamma+AB$, it follows that $\varepsilon W+u>0$ in D. In an analogous way one can show that $\varepsilon W-u>0$ in D. From the inequalities $\varepsilon W+u>0$ and $\varepsilon W-u>0$ in D, by virtue of the arbitrariness of ε, it follows that $u\equiv 0$.

Let us prove existence. By direct verification it is easy to make sure that if $v(x,y)$ is a solution of the equation

$$L_1(v) = yv_{yy} + v_{xx} + \left(a + 2y\frac{W_y}{W}\right) v_y + \left(b + 2\frac{W_x}{W}\right) v_x + \frac{L(W)}{W} v = 0, \quad (9)$$

satisfying the boundary condition

$$v = \frac{\omega f}{W} = \varphi \quad \text{on } \Gamma + AB, \quad (10)$$

then $u = \varepsilon W v$ will be a solution of our problem.

Obviously, φ is a continuous function on $\Gamma + AB$. Following (1), it is easy to show that there exists a bounded solution of equation (9) which satisfies the boundary condition (10).

As is known (1), in order for the boundary conditions to be satisfied, it is sufficient to construct, at each point of $\Gamma + AB$, a so-called barrier. Obviously, barriers exist at the points of Γ, while at the points of the segment AB one may take as a barrier the function

$$F(x,y) = (-\log y)^{-\sigma} + (x - x_0)^2, \qquad 0 < \sigma < \alpha.$$

2. Denote by G the set of those points of the segment AB where

$$\lim_{y \to 0} \omega(x,y) = \infty,$$

and let $G_0 = \Gamma + AB - G$.

Theorem 2. *If $u(x,y)$ is a twice continuously differentiable solution of equation (1) in D, and its boundary values are continuous on the closed set $\overline{G_0}$, while on G*

$$\lim_{y \to 0} \frac{u}{\omega} = 0,$$

then such a solution is bounded in $\overline{D}$, and it is uniquely determined by specifying continuous data on Γ and on those pieces of the segment AB where $a(x,0) < 1$.

Let us prove boundedness. Consider the function

$$\Phi = \varepsilon W + M - (x - \delta)^p + u.$$

It is easy to see that, by choosing M and p, for any $\varepsilon > 0$ one can ensure that $\Phi > 0$ on $\Gamma + AB$ and $L(\Phi) < 0$ in D; hence it follows that $\Phi > 0$ in D, i.e.

$$\varepsilon W + M - (x+\delta)^p + u > 0.$$

Similarly, one can obtain

$$\varepsilon W + M - (x+\delta)^p - u > 0.$$

From these two inequalities, by virtue of the arbitrariness of $\varepsilon > 0$, it follows that u is bounded in $\overline{D}$. From the boundedness of u, on the basis of (1), follows the existence of a solution of equation (1) which assumes the prescribed values on Γ and on those pieces of the segment AB where $a(x,0) < 1$.

The uniqueness of such a solution is proved analogously to the proof of Theorem 1.

Theorems 1 and 2 remain valid also for the equation in n-dimensional space

$$x_n \frac{\partial^2 u}{\partial x_n^2} + \sum_{i=1}^{n-1} \frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^{n} a_i(x_1, \dots, x_n) \frac{\partial u}{\partial x_i} + c(x_1, \dots, x_n) u = 0, \qquad c \leq 0,$$

for a bounded domain D lying in the half-space $x_n > 0$ and resting on the hyperplane $x_n = 0$; a_i and c are analytic functions for finite values $x_1, x_2, \dots, x_{n-1}$, $x_n \geq 0$.

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1. M. V. Keldysh, DAN, 77, No. 2 (1951).

2. A. V. Bitsadze, Proceedings of the 3rd All-Union Mathematical Congress, 3 (1957), in press.

Note: Figure translations are in progress. See original paper for figures.