GEOPHYSICS

1957

SovietRxiv

Abstract

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GEOPHYSICS

L. N. GUTMAN

A THEORETICAL MODEL OF A CUMULUS CLOUD

(Presented by Academician A. A. Dorodnitsyn, 5 X 1956)

Let us consider, in the coordinate system (x, z) (x is the horizontal coordinate, z the coordinate directed upward), the plane stationary problem of ordered thermal convection caused by the vertical instability of the atmosphere. Assuming that there is no general motion of the air, we shall simplify the complete system of equations of atmospheric thermohydrodynamics by virtue of the smallness of the perturbations of temperature and pressure $\vartheta(x, z)$ and $p(x, z)$ in comparison with $\theta(z)$ and $P(z)$—the temperature and pressure in the atmosphere at a sufficiently large distance from the region of convection (we assume that $\theta(z)$ and $P(z)$ are prescribed functions satisfying the equation of statics). Then, neglecting the influence of variation in the horizontal direction of the relative humidity, as well as turbulence and the Coriolis force, we shall have*:

$$\frac{du}{dt} = -R\theta\frac{\partial}{\partial x}\left(\frac{p}{P}\right); \quad \frac{dw}{dt} = -R\theta\frac{\partial}{\partial z}\left(\frac{p}{P}\right) + \lambda\vartheta; \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = \sigma w; \quad \frac{d}{dt} = u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}; \tag{1}$$

$$\frac{d\vartheta}{dt} = \alpha(z)w; \quad \alpha(z) = \begin{cases} \gamma - \gamma_a, & \text{for } f < 100\%, \\ \gamma - \gamma_b(\theta, P), & \text{for } f = 100\%. \end{cases} \tag{2}$$

Here u, w are the horizontal and vertical components of the wind velocity; R is the gas constant for air; f is the relative humidity; $\gamma_a, \gamma_b(\theta, P)$ are the dry- and moist-adiabatic gradients, the latter corresponding to the unperturbed state of the atmosphere; $\gamma = -d\theta/dz$; $\lambda = g/\theta$; $\sigma = (g - R\gamma)/R\theta$. For simplicity we take $\lambda = \text{const}$ and $\sigma = \text{const}$, thereby allowing an error of order 5%. From the system (1), (2) the air density ρ has been eliminated by means of the Clapeyron equation.

The natural assumption of symmetry of the pattern of motion and of the local character of the perturbations leads to the following boundary conditions in x:

$$u = 0, \quad w < \infty \quad \text{for } x = 0; \qquad w = \vartheta = p = 0 \quad \text{for } x = \infty. \tag{3}$$

It is easy to see that the nonlinear homogeneous system (1), (2), with homogeneous boundary conditions (3), has the zero solution, corresponding, evidently, to the equilibrium state of the atmosphere. It will be shown below that if, even in a very thin layer of the atmosphere, $\alpha(z) > 0$ (a layer of instability), then, along with the zero solution, there will exist a certain nonzero solution, corresponding, apparently, to the process of release—

* The assumption of stationarity is quite admissible, since it can be shown that the process under study is quasi-stationary in essence (the time of establishment of the motion is of the order of several minutes). The neglect of turbulence and of the Coriolis force is due to the scales of the motion, which are assumed both horizontally and vertically to be of the order of several kilometers.

of instability, i.e., to a process which, under certain conditions, may lead to the formation of a cumulus cloud.

Introducing the stream function ψ and the vorticity ω from the relations:

$$u = -e^{\sigma z}\frac{\partial \psi}{\partial z}; \quad w = e^{\sigma z}\frac{\partial \psi}{\partial x}; \quad \omega = e^{2\sigma z}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} + \sigma \frac{\partial \psi}{\partial z}\right), \tag{4}$$

we obtain, instead of (1), (2),

$$\frac{\partial(\psi, \vartheta)}{\partial(x, z)} = \alpha(z)\frac{\partial \psi}{\partial x}; \quad \frac{\partial(\psi, \omega)}{\partial(x, z)} = \lambda \frac{\partial \vartheta}{\partial x}\ ^{*}. \tag{5}$$

If now $\alpha(z)$ is represented in the form $\alpha = \alpha_0(1 + \alpha_1 z + \alpha_2 z^2 + ...)$, where α_k are numerical coefficients, then it can be shown that (5) and the last equation from (4), which constitute a closed system, will have a solution of the form **

$$\psi = \frac{z^2}{2}\sqrt{\alpha_0 \lambda a}\,(\psi_0 + z\psi_1 + z^2\psi_2 + ...), \quad \vartheta = \alpha_0 z(\vartheta_0 + z\vartheta_1 + ...), \tag{6}$$

$$\omega = \frac{\sqrt{\alpha_0 \lambda a}}{2}(\omega_0 + z\omega_1 + ...),$$

where ψ_n, ϑ_n, and ω_n are functions of only one argument $\varphi = \operatorname{arctg} x/z$, ($a =$ const), satisfying the following infinite system (a prime denotes derivatives with respect to φ):

$$2\psi_0\vartheta_0' + \psi_0'\vartheta_0 + \psi_0' = 0; \quad a\psi_0\omega_0' + 2\vartheta_0' = 0;$$

$$2\psi_0\vartheta_1' + 3\psi_1\vartheta_0' - 2\psi_0'\vartheta_1 - \psi_1'\vartheta_0 + \psi_1' + \alpha_1\psi_0' = 0; \tag{7}$$

$$a(2\psi_0\omega_1' + 3\psi_1\omega_0' - \psi_0'\omega_1) + 4\vartheta_1' = 0;$$

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and

$$\omega_0 = L_0; \quad \omega_1 = L_1 + \sigma(2L_0 + 2\psi_0 - \sin\varphi\cos\varphi\,\psi_0'); \dots \tag{8}$$

where it is denoted:

$$L_n(\varphi) = \cos^2\varphi\,\psi_n'' - (n+2)\sin 2\varphi\,\psi_n' + (n+1)(n+2)\psi_n. \tag{9}$$

The system (7), (8) must be solved under the boundary conditions $\psi_n(0) = 0$, $\psi_n(\pi/2) < \infty$, $\vartheta_n(\pi/2) = 0$, which are a consequence of (3). We place the origin of coordinates on the lower boundary of the instability region, for example, at the condensation level, and thereby ensure fulfillment of the inequality $\alpha_0 > 0$. We shall show that for $\alpha_0 > 0$, in addition to the trivial solution there exists a certain nonzero solution satisfying conditions (3) in x and, in z, the conditions: $u = w = \vartheta = p = 0$ for $z \le 0$, which follow from (6).

Successively integrating equations (7), taking into account the conditions $\vartheta_n(\pi/2) = 0$, $\psi_0(\pi/2) = 1$, we find ***

$$\vartheta_0 = 1 - \sqrt{\psi_0}; \quad \vartheta_1 = \frac{1}{2}\left[(a_1 - \alpha_1)\psi_0 + \alpha_1 - \frac{\psi_1}{\sqrt{\psi_0}}\right]; \dots \tag{10}$$

$$\omega_0 = \frac{1}{a}\left(c_0 - \frac{2}{\sqrt{\psi_0}}\right); \quad \omega_1 = \frac{1}{a}\left[2(a_1 - \alpha_1) + c_1\sqrt{\psi_0} + \frac{\psi_1}{\psi_0^{3/2}}\right]; \dots \tag{11}$$

where

$$a_n = \psi_n(\pi/2); \quad c_0 = 2(a+1); \quad c_1 = 2\alpha_1 + 6\sigma a - 3(1-2a)a_1; \dots$$

* In this equation, for simplicity, the small term $R\gamma d\left(\frac{p}{P}\right)/dx$ has been omitted. Such omission introduces a

** The presence of first integrals of system (5) makes it possible to estimate the error arising when the series

*** It can be shown that each of the equations must, after integration, lead to some algebraic relation betwee

Bearing in mind the need to construct integral equations for determining ψ_n, we solve equation (9) with respect to ψ_n, taking the left-hand side as known. Choosing the bounded solution, we obtain:

$$\psi_n = D_n^{\varphi}(L_n) = \frac{\sec^{n+2}\varphi}{n+2}\int_{\varphi}^{\pi/2} \sin[(n+2)(\chi-\varphi)]\cos^n\chi L_n(\chi)\,d\chi. \qquad (12)$$

Taking into account that L_n are given by formulas (8), (11), we see that (12) constitutes integral equations with respect to $\psi_n(\varphi)$. Thus, substituting L_0 from (8), (11) into (12), we obtain

$$\psi_0 = 1 - \frac{1}{a} D_0^{\varphi}\left(\frac{2}{\sqrt{\psi_0}} - 2\right), \qquad (13)$$

where the condition $\psi_0(0) = 0$ leads to the expression

$$a = D_0^0\left(\frac{2}{\sqrt{\psi_0}} - 2\right). \qquad (14)$$

The integral equations for the remaining ψ_n include the dimensional parameters σ and α_k. In order to avoid the need to solve the integral equations for each particular combination of values of the dimensional parameters, we pass to a system of functions $\psi_{i,j}(\varphi)$ from the relations:

$$\psi_1 = \alpha_1\psi_{11} + \sigma\psi_{12}; \quad \psi_2 = \alpha_2\psi_{21} + \alpha_1^2\psi_{22} + \alpha_1\sigma\psi_{23} + \sigma^2\psi_{24}; \ldots$$

For $\psi_{i,j}(\varphi)$ we shall have a sequence of dimensionless integral equations, the solution of which, just as the solution of the integral equations (13), (14), can be carried out by the method of successive approximations once and for all. By the indicated procedure we tabulated the first 6 functions $\psi_{i,j}(\varphi)$. Their graphs are shown in Fig. 1. Having $\psi_n(\varphi)$, from (1), (4), (10) it is easy to find u, w, ϑ, and p.

Fig. 1: Graphs of the first 6 functions $\psi_{i,j}(\varphi)$.

The solution obtained also makes it possible to determine $V(x, z)$—the rate of formation of the liquid phase. Indeed, with the accuracy adopted in the problem, from (1) we obtain:

$$V(x,z) = \mu(z)w, \quad \text{where} \quad \mu(z) = \begin{cases} 0, & \text{for } f < 100\%, \\ \dfrac{c_p}{L(\theta)}\left[\gamma_a - \gamma_b(\theta, P)\right], & \text{for } f = 100\%, \end{cases}$$

where $L(\theta)$ is the latent heat of condensation or sublimation, and c_p is the heat capacity of air.

If we assume that the incoming droplet-liquid moisture does not fall out as precipitation, but is redistributed in space by air currents, then, neglecting evaporation, we shall have the following equation for $W(x,z)$—the water content:

$$u\frac{\partial}{\partial x}\left(\frac{W}{\rho}\right) + w\frac{\partial}{\partial z}\left(\frac{W}{\rho}\right) = V = \mu(z)w. \tag{15}$$

It follows immediately that

$$W = \rho\left[f_1(\psi) + \int_0^z \mu(z)\,dz\right],$$

where the form of the function f_1 is determined from the boundary condition of the problem $W = 0$ at $x = \infty$ (of course, this condition can be satisfied only on those branches of the streamlines along which air flows into the region of convection). The distribution of specific humidity can likewise be found from an equation very similar to (15).

The calculations we have carried out allow us to conclude that the hydrodynamic picture corresponding to the solution obtained is very reminiscent of a cumulonimbus cloud. To illustrate this conclusion, let us consider a specific example.

Suppose that at some height above the Earth' s surface there is a dry-unstable layer 0.5 km thick. Immediately above it, in a layer 0.5 km thick, the air is stratified moist-unstably. Finally, let there be a temperature inversion above this layer. Then one may take: $\alpha_0 = 3$ deg/km; $\alpha_1 = -1\ \text{km}^{-1}$; $\alpha_2 = -0.1\ \text{km}^{-2}$. The quantity α_2 has been assigned a negative value in order to strengthen the "inversion."

Fig. 2: Streamlines calculated by formula (6).

Fig. 2

The streamlines calculated by formula (6) (the right half of the cloud) are shown in Fig. 2. The cloud boundary is drawn with a dashed line. The cloud base *ab* evidently lies at the condensation level. The lateral boundary *bcd* is drawn along the isoline $w \simeq 0.1$ m/sec. Finally, the upper boundary *de* corresponds to the isoline $w \simeq 0$. We see that the upper part of the cloud resembles an "anvil."

It should be noted that with a weaker inversion ($\alpha_2 = 0$) the "anvil" does not appear.

It is also seen from Fig. 2 that the entire cloud is encompassed by ascending air motion, and descending motions are observed neither in the cloud nor outside it. Hence one may conclude that in nature the compensating subsidence of air masses must occur very slowly and over a large area (in plan), considerably exceeding the area of the cloud itself. The largest values of the quantities w, ϑ, ρ, and W, occurring at different z near the axis of symmetry of the cloud, are as follows (z is given in km): $w_{z=1.5} = 5$ m/sec; $\vartheta_{z=1} = 1.5^{\circ}$; $\rho_{z=1} = 0.3$ mb; $W_{z=1} = 2$ g/m^3. In calculating W we took $\gamma_a - \gamma_b = 5 \cdot 10^{-3}$ deg $\cdot$ m^{-1}. Let us now suppose that the length of the cloud in the direction of the y-axis is 3 km. In this case, as is easy to calculate, about 10^7 tons of air per hour enter through the lower base of the cloud; along with this, the cloud passes through itself considerably more air, namely $5 \cdot 10^7$ tons per hour. Thus, the supply of moist air to the given cloud is effected mainly through the inflow of air from the sides. As a result of condensation of water vapor in our cloud, $2 \cdot 10^4$ tons of droplet-liquid moisture are formed per hour. If all the moisture formed falls to the ground, this can give, averaged over the area, up to 2.3 mm of precipitation per hour.

In conclusion I express my deep gratitude to Corresponding Member of the Academy of Sciences of the USSR I. A. Kibel' for valuable advice, and also to V. M. Bowsheverov, discussions with whom contributed to the formulation of the work.

Institute of Atmospheric Physics
Academy of Sciences of the USSR

Received
4 X 1956

Note: Figure translations are in progress. See original paper for figures.