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Abstract

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EMBEDDING THEOREMS FOR ABSTRACT FUNCTIONS OF SETS

In a previous note ([1]) we constructed a space Ψ_p of abstract functions of sets, which is the closure, in the sense of the norm $\| \|_{\Phi_p}$, of the set of step measurable functions. Generally speaking, Ψ_p is a proper subspace of Φ_p and consists of functions that are absolutely continuous and continuous under shifts.

Some embedding theorems, which have found applications in the theory of partial differential equations ([2,3]), can be extended to the elements of Ψ_p. We shall show how this can be done.

Define the integrals

$$\int \omega(P)\, d\varphi(E), \tag{1}$$

where the function $\varphi(E) \in \Psi_p$, and $\omega(P)$ is an element of $L_{p'}$. (For $p = 1$, $\omega(P)$ must be regarded as a bounded measurable function.) The integral (1) always exists and is an element of X, as is proved by means of a passage to the limit from step functions.

We shall study integrals

$$U(Q) = \int \omega(Q, P)\, d\varphi(E), \tag{2}$$

which take values in X, where $\omega(P, Q)$ is an abstract function of the point Q, whose values are elements of $L_{p'}$ as functions of the point P.

Theorem 1. *Let $\varphi(E) \in \Psi_p$, and let $\omega(Q, P)$ be continuous as a function of the point $Q \in \Omega$. Then $U(Q)$ will be a continuous abstract function of the point Q.*

The proof of this theorem is carried out by direct computation of

$$\|U(Q + \Delta Q) = U(Q)\|_X. \tag{3}$$

As an application of this theorem we give Theorem 2.

Theorem 2 (on integrals of potential type). *The integral*

$$U(Q) = \int \frac{K(P,Q)}{r^{\lambda}}\, d\varphi(E), \tag{4}$$

where $K(P,Q)$ is a bounded function of the pair of points P, Q from R_n, continuous in each of them when $P \neq Q$; r is the distance from P to Q; $\lambda < n/p'$, and $\varphi(E) \in \Phi_p$, is a continuous abstract function of the point Q, taking values in X.

Indeed, the kernel $\omega(Q,P) = K(P,Q)/r^{\lambda}$ satisfies the conditions of the preceding theorem, whence our assertion follows.

Consider in Euclidean m-dimensional space $R_m(y_1, y_2, \dots, y_m)$ some s-dimensional linear manifold S_s, which we shall specify by the equations $y_{s+1} = y_{s+2} = \dots = y_m = 0$, together with a system of parallel manifolds $S_s(y_{s+1}, \dots, y_m)$ corresponding to constant values $y_{s+1}, \dots, y_m$. Let, in some domain Ω of the variables $y_1, y_2, \dots, y_s$, there be given an abstract function $\varphi(E_s, y_{s+1}, \dots, y_m \mid P)$ of sets $E_s \subset \Omega$, depending also on the variables $y_{s+1}, y_{s+2}, \dots, y_m$, and taking values in $L_{p'}$ as a function of $P \in R_n$. Suppose that this function is continuous with respect to E_s and continuous with respect to translation in m-dimensional space (this notion needs no explanation).

Theorem 3. The abstract function

$$U(E_s, y_{s+1}, \dots, y_m) = \int \omega(E_s, y_{s+1}, \dots, y_m \mid P)\, d\varphi(E), \tag{5}$$

where $\varphi(E) \in \Psi_p$, is a function with values in X, absolutely continuous with respect to E_s and continuous with respect to translation in $R_m(y_1, \dots, y_m)$.

The proof of Theorem 3 is also obvious.

As an application of Theorem 3 we indicate Theorem 4.

Theorem 4. The integral

$$U(E_s, y_{s+1}, \dots, y_n) = \left[\iint_{E_s} \frac{K(Q_s, y_{s+1}, \dots, y_n, P)}{r^{\lambda}}\, dQ_s \right] d\varphi(E), \tag{6}$$

where $\varphi(E) \in \Psi_p$, $n/p' < \lambda < s/q$, and $K(Q,P)$ is a bounded function of P and Q, continuous for $P \neq Q$, is an abstract function of $E_s, y_{s+1}, \dots, y_n$, absolutely continuous and continuous with respect to translation, with values in X.

Theorem 4 is reduced to Theorem 3 if one observes that the kernel

$$\omega(E_s, y_{s+1}, \dots, y_n \mid P) = \int_{E_s} \frac{K(Q_s, y_{s+1}, \dots, y_n \mid P)}{r^{\lambda}}\, dQ_s \tag{7}$$

satisfies the conditions of the preceding theorem. This last assertion has essentially been proved, for example, in the author' s book ([2]).

Let us introduce the notion of a derivative of an abstract function of sets. Let $\psi_{\alpha_1\cdots\alpha_n}(E)$ be such that for any continuously differentiable k-times finite function $\omega(P)$ the identity

$$\int \omega \, d\psi_{\alpha_1\cdots\alpha_k}(E) = (-1)^{\alpha_1+\alpha_2\cdots+\alpha_k} \int \frac{\partial^k \omega}{\partial x_1^{\alpha_1}\cdots\partial x_k^{\alpha_k}} \, d\varphi(E). \tag{8}$$

Then we shall say that

$$\psi_{\alpha_1\cdots\alpha_n} = \frac{\partial^k \varphi}{\partial x_1^{\alpha}\cdots\partial x_{n_1}^{\alpha_n}}. \tag{9}$$

The functions $\varphi(E)$ for which all derivatives of order l belong to Φ_p form the space $\Phi_p^{(l)}$. As the norm in this space $\|\varphi\|_{\Phi_p^{(l)}}$ it is convenient to take, for example, the sum of the norms in Φ_p of all its derivatives of order l and the norm of $\varphi(E)$ itself in Φ_1:

$$\|\varphi(E)\|_{\Phi_p^{(l)}} = \|\varphi(E)\|_{\Phi_1} + \sum \left\| \frac{\partial^l \varphi}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} \right\|_{\Phi_p}.$$

Theorem 5. If $lp > n$, every element $\varphi(E)$ of $\Psi_p^{(l)}$ is a continuous function of the point $\mathbf{Q}$. The modulus of continuity of this function is equal to $\delta(\varepsilon) = \varepsilon^\beta$, where $\beta = \min\left(1-0,\ l - \frac{n}{p}\right)$.

The inequality

$$\|\varphi(E)\|_C = \max_E \|\varphi(E)\|_X \le A\|\varphi(E)\|_{\Phi_p^{(l)}}.$$

is valid.

Theorem 6. If $lp < n$, every element $\varphi(E)$ of $\Psi_p^{(l)}$ on any manifold of dimension $s > n - lp$ is an element of Ψ_q, where $\frac{s}{q} \le \frac{n}{p} - l$. The function $\varphi(E_s)$ on such manifolds is continuous with respect to translation. Its modulus of continuity is equal to $\delta(\varepsilon) = \varepsilon^\beta$, where $\beta = \min\left(1-0,\ l - \frac{n}{p} - \frac{s}{q} - 0\right)$ when $\frac{s}{q} < \frac{n}{p} - l$.

The inequality

$$\|\varphi(E_s)\|_{\Phi_q} \le A\|\varphi(E)\|_{\Phi_p^{(l)}}.$$

is valid.

The proof of the theorems stated is based on passing to mean functions. As was noted in the preceding note ([1]), these functions are continuous functions of a point. For the mean functions $\varphi_h(E)$, in the abstract case the identity remains valid

$$\varphi_h(\mathbf{Q}) = \int K_0(\mathbf{Q},\mathbf{P})\varphi_h(\mathbf{P})\,dP + \sum \int K_{\alpha_1\cdots\alpha_n}(\mathbf{Q},\mathbf{P}) \frac{\partial^l \varphi_h(\mathbf{P})}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}\,dP, \quad (10)$$

which expresses the value $\varphi_h(\mathbf{P})$ in terms of derivatives and is usually used to prove embedding theorems for numerical functions ([2]).

From identity (10) our theorems are obtained by applying the theory of integrals of potential type. The limiting passage then yields the corresponding theorem for functions of sets.

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CITED LITERATURE

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[2] S. L. Sobolev, Mat. sborn., **4** (46), 3 (1938).
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[4] S. L. Sobolev, Mat. sborn., **5** (47), 1 (1939).

Note: Figure translations are in progress. See original paper for figures.