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Abstract

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SOME THEOREMS ON THE STABILITY OF SOLUTIONS OF PARABOLIC SYSTEMS

(Presented by Academician I. G. Petrovskii on 14 II 1957)

In the present note we set forth some theorems on the Lyapunov stability of solutions of systems parabolic in the sense of I. G. Petrovskii ([1])

$$\frac{\partial u}{\partial t} = P\left(t, \frac{1}{i}\frac{\partial}{\partial x}\right) u \equiv \sum_{k=0}^{2b} P_k\left(t, \frac{1}{i}\frac{\partial}{\partial x}\right) u \tag{1}$$

$$\left(P_k\left(t, \frac{1}{i}\frac{d}{dx}\right)\right.$$

is a differential operator of order k with continuous (for $t \geq 0$) coefficients, defined in the half-space $t \geq 0$ and belonging in each strip $[0, T]$ to the class E of uniqueness of the solution of the Cauchy problem. These results are obtained by studying the Green matrix of the system (1) in the half-space $t \geq 0$, which is analogous to the investigation of this matrix carried out by the author for the establishment of various Liouville theorems. Some of the criteria obtained in this way are direct generalizations of the well-known theorem of A. M. Lyapunov ([2]) to partial differential equations of parabolic type. We note that parabolicity of the system (1) is assumed in each strip $[0, T]$, $0 < T < \infty$. The question studied is that of the stability of the trivial solution of system (1).

Definition 1. The Green matrix $G(t, \tau, x)$ of system (1) satisfies condition $\mathcal{L}_1$ ($\mathcal{L}_2$) if, for any $t > \tau \geq 0$ and $x_1, \ldots, x_n$,

$$|G(t, \tau, x)| \leq C_0 a(t, \tau)^{-n} \exp\left\{-C_0 \left|\frac{x}{a}\right|^q\right\}; \qquad q = \frac{2b}{2b-1}; \tag{$\mathcal{L}_1$}$$

$$|G(t, \tau, x)| \leq C_0 a(t, \tau)^{-n} \exp\left\{-\delta_0 a(t, \tau)^{2b} - c_0 \left|\frac{x}{a}\right|^q\right\}; \tag{$\mathcal{L}_2$}$$

C_0, δ_0, c_0 are positive constants; $a(t, t) = 0$; $a(t, \tau)$ is a monotonically increasing function of the argument t; in $\mathcal{L}_2$, $a(t, \tau) \to \infty$ as $t \to \infty$ for fixed τ.

Definition 2. The trivial solution of system (1) is called **Lyapunov stable** if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, for every solution $u(x, t) \in E$

such that $|u(x,0)| \leq \delta e^{k|x|}$, the inequality $|u(x,t)| \leq \varepsilon e^{k|x|}$ holds (E_k-stability). The trivial solution is asymptotically stable if, in addition, $|u(x,t)|e^{-k|x|} \to 0$ uniformly in x as $t \to \infty$ (E_k-asymptotic stability).

From the estimates $\mathcal{L}_1, \mathcal{L}_2$ and the representation of the solution by means of the Green matrix it follows:

Theorem 1. *If condition $\mathcal{L}_1$ is fulfilled, then the trivial solution of system (1) is E_0-stable. If condition $\mathcal{L}_2$ is fulfilled, then the trivial solution is E_k-asymptotically stable, $k < (2b\delta_0)^{1/2b}(c_0 q)^{1/q}$.*

Condition $_1$ is fulfilled if:

1) a)

$$R(P(t,\sigma)a,a) \leq \{-f_1(t)|\sigma|^{2b} + f_2(t)\}|a|^2$$

for any complex-valued vector a, any $t \geq 0$, and any real $\sigma_1, \dots, \sigma_n$; (,) denotes the scalar product; $f_l(t)$ are positive functions;

$$\int_0^\infty f_l(t)\,dt < +\infty, \quad l = 1, 2;$$

b) the modulus of the quotient obtained by dividing the coefficients of the system by $f_1(t)$ is bounded.

2)

$$R(P(t,\sigma)a,a) = \sum_{k=2r}^{2b} R(P_k(t,\sigma)a,a) \leq \{-f_1(t)|\sigma|^{2b} + |\sigma|^{2r} + f_2(t)\}|a|^2;$$

$f_1(t), f_2(t)$ are positive functions;

$$\int_0^\infty f_2(t)\,dt < +\infty;$$

condition 1b).

3) The system has constant coefficients;

$$P(\sigma) = \sum_{k=2r}^{2b} P_k(\sigma);$$

the equation $\det\{P(\sigma) - \lambda E\} = 0$ has a root with real part tending to zero only when $\sigma = 0$; the matrix $P_{2r}(\sigma)$, for $|\sigma| = 1$, has eigenvalues whose real parts do not tend to zero for any $\sigma_1, \dots, \sigma_n$.

Condition $_2$ is fulfilled if:

4)

$$R(P(t,\sigma)a,a) \leq -f_1(t)\big[|\sigma|^{2b} + \delta_0^*\big]|a|^\theta; \quad \int_\tau^t f_1(\beta)\,d\beta = a^{2b}(t,\tau) \to \infty \quad (t \to \infty);$$

$$\delta_0 < \delta_0^*;$$

condition 1b).

5) The system has constant coefficients; the equation

$$\det\{P(\sigma) - \lambda E\} = 0$$

has roots whose real parts do not tend to zero for any real $\sigma_1, \dots, \sigma_n$;

$$a(t, \tau) = (t - \tau)^{1/2b};$$

$$\delta_0 < \min\{-R\lambda_k^*\};$$

λ_k^* are the roots of the equation

$$\det\left\{\sum_{k=0}^{2b} P_k\left(\frac{\sigma}{|\sigma^*|}\right)\left(\frac{1}{|\sigma^*|}\right)^{2b-k} - \lambda^* E\right\} = 0; \quad |\sigma^*| = \sqrt{|\sigma|^2 + 1}.$$

Definition 3. The trivial solution of system (1) is called **unstable** if there exists an $\varepsilon_0 > 0$ such that, no matter what $\delta > 0$ is chosen, there is always a solution of system (1) $u(x,t) \in E$, $|u(x,0)| < \delta e^{k|x|}$, and a $T > 0$ such that

$$\sup_x\{|u(x,T)|e^{-k|x|}\} > \varepsilon_0.$$

Theorem 2. 1) The trivial solution of system (1) will be E_0-unstable if, for each vector

$$a = (c_1, c_j, b_{j+1}, \dots, b_{j+m}, c_{j+m+1}, \dots, c_N),$$

$$b = (0, \dots, 0, b_{j+1}, \dots, b_{j+m}, 0, \dots, 0), \quad c = a - b, \quad m \geq 1,$$

and for at least one set $\sigma_1^*, \dots, \sigma_n^*$,

$$R\{(P(t, \sigma^*)a, b) - (P(t, \sigma^*)a, c)\} \geq f(t)|b|^2; \quad \int_0^\infty f(t)\,dt = +\infty.$$

2) For a system with constant coefficients the trivial solution will be E_0-unstable if, for at least one set $\sigma_1^*, \dots, \sigma_n^*$, at least one root of the determinant of the matrix $P(\sigma) - \lambda E$ has positive real part.

Remark. For the system

$$\frac{\partial u}{\partial t} = P_{2b}\left(\frac{1}{i}\frac{\partial}{\partial x}\right)u$$

the trivial solution will be E_0-stable, but E_η-unstable for any $\eta > 0$. For the equation

$$\frac{\partial u}{\partial t} = \Delta u - u; \quad C_0 = \frac{1}{4}; \quad \delta_0 = 1; \quad 2b = 2,$$

therefore $k < 1$, the trivial solution is $E_{1-\eta}$-asymptotically stable, but $E_{1+\eta}$-unstable, since this equation has the solution $\delta e^{(\eta^2+2\eta)t+(\eta+1)x}$, $\eta > 0$.

If one considers systems parabolic in the sense of I. G. Petrovskii and containing derivatives of higher order with respect to t, then already for systems with constant coefficients containing only the group of terms of highest order of differentiation (in the parabolic sense) the stability theorems will be false, since the solutions of such systems will be powers of t. However, the stability criteria 4), 5) remain valid if in them, instead of $P(t,\sigma)$, one puts $|\sigma|^{2b}P^*(t,\sigma)$, where the latter matrix corresponds to the first-order-in-t system obtained from the original higher-order system, and is obtained from the matrix of the corresponding system of ordinary differential equations by the transformation of I. G. Petrovskii ([1], pp. 30–31). We shall give some easily proved stability theorems for the system obtained from system (1) by adding a nonlinear term.

Consider the system

$$\frac{\partial u}{\partial t} = P\left(t, \frac{1}{i}\frac{\partial}{\partial x}\right) u + f(t,x,u), \tag{2}$$

and suppose that it has the trivial solution.

Theorem 3. 1) If the following conditions are satisfied: a) for the linear system, condition Π_2 with an additive function $a^{2b}(t,\tau)$, i.e. $a^{2b}(t,\tau) = a^{2b}(t,\beta) + a^{2b}(\beta,\tau)$, $\tau < \beta < t$; b) $f(t,x,u)$ is continuous jointly in its arguments for $t \geq 0$, $-\infty < x_s < \infty$, $s = 1,2,\dots,n$, $-\infty < u_k < \infty$, $k = 1,2,\dots,N$; c) $|f(t,x,u)| \leq g(t)|u|$ for sufficiently small

$$|u|e^{-k|x|}; \quad \alpha a^{2b}(t,0) - \int_0^t g(\tau)\,d\tau \to \infty, \quad t \to \infty;$$

α is some positive constant determined by δ_0, C_0, k, b, then the trivial solution of system (2) is E_k-asymptotically stable.

2) If the following conditions are satisfied: a) for the linear system, condition Π_1; b) $f(t,x,u)$ is continuous jointly in its arguments for $t \geq 0$, $-\infty < x_s < \infty$, $s = 1,2,\dots,n$, $|u| \leq k_0$; c) $|f(t,x,u)| \leq g(t)|u|$ for sufficiently small $|u|$;

$$\int_0^\infty g(\tau)\,d\tau < +\infty,$$

then the trivial solution of system (2) is E_0-stable.

From part 1) of Theorem 3 it follows:

Theorem 4. If the following conditions are satisfied: a) for the linear system, condition Π_2 with $a(t,\tau) = (t-\tau)^{1/2b}$; b) condition b) of part 2) of Theorem 3; c) $|f(t,x,u)| \leq C|u|^{1+\beta}$, $\beta > 0$, for sufficiently small $|u|$, then the trivial solution of system (2) is asymptotically E_0-stable.

In particular, condition a) of Theorem 4 is satisfied for systems with constant coefficients for which the roots of the determinant of the matrix $P(\sigma) - \lambda E$ have negative real parts for all real $\sigma_1, \dots, \sigma_n$. If one considers f independent of x, and studies the stability of the trivial solution of system (2) in the class of solutions depending only on t, then Theorem 4 coincides with the well-known theorem of A. M. Lyapunov ([2]), and part 1) of Theorem 3 ($k = 0$) with the proposition proved in ([3]) (p. 234).

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.