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## Abstract

## Full Text

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# ON THE NUMERICAL SOLUTION OF A BOUNDARY-VALUE PROBLEM FOR A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

*(Presented by Academician S. L. Sobolev on 24 IX 1956)*

Let the problem be given

$$a_i \frac{d^2 u_i}{dx^2} + \sum_{j=1}^{n} b_{i,j} u_j + \lambda \sum_{j=1}^{n} c_{i,j} u_j = 0, \quad x \in (0, R), \quad x \neq r_k; \tag{1}$$

$$u_i(0) = u_i(R) = 0; \tag{2}$$

$$u_i(r_k - 0) = u_i(r_k + 0); \tag{3}$$

$$a_i(r_k + 0) \frac{du_i(r_k + 0)}{dx} - a_i(r_k - 0) \frac{du_i(r_k - 0)}{dx} = \frac{a_i(r_k + 0) - a_i(r_k - 0)}{r_k} u_i(r_k), \tag{4}$$

$$0 = r_0 < r_1 < \dots < r_k < \dots < r_{s-1} < r_s = R,$$

$$k = 1, 2, \dots, s - 1,$$

$$i = 1, 2, \dots, n,$$

where the unknowns are the vector-functions $u\{u_1, u_2, \dots, u_n\}$ and the scalar $\lambda$; the real coefficients $a_i, b_{i,j}$, and $c_{i,j}$ $(i, j = 1, 2, \dots, n)$ may have discontinuities of the first kind at the points $r_k$ $(k = 1, 2, \dots, s - 1)$.

Divide $[r_{k-1}, r_k]$ into $m_k$ parts; denoting the resulting step by $h_k$ $(h_k = (r_k - r_{k-1})/m_k,\ k = 1, 2, \dots, s)$, we approximate the original problem (1)–(4) by the following algebraic one:

$$a_{i,k}\frac{\Delta^2 u^{(h)}_{i,k,j}}{h_k^2}+\sum_{p=1}^{n} b_{i,p,k,j}u^{(h)}_{p,k,j}+\lambda^{(h)}\sum_{p=1}^{n} c_{i,p,k,j}u^{(h)}_{p,k,j}=0, \tag{5}$$

$$j=1,2,\dots,m_k-1,\quad k=1,2,\dots,s,\quad i=1,2,\dots,n;$$

$$u^{(h)}_{i,1,0}=u^{(h)}_{i,s,m_s}=0,\quad i=1,2,\dots,n; \tag{6}$$

$$a_{i,k+1}\frac{\Delta u^{(h)}_{i,k,m_k}}{h_{k+1}}-a_{i,k}\frac{\Delta u^{(h)}_{i,k,m_k-1}}{h_k}=\frac{a_{i,k+1}-a_{i,k}}{r_k}\,u^{(h)}_{i,k,m_k}, \tag{7}$$

$$k=1,2,\dots,s-1,\quad i=1,2,\dots,n,$$

where

$$\Delta\varphi_j=\varphi_{j+1}-\varphi_j;\qquad \Delta^2\varphi_j=\varphi_{j-1}-2\varphi_j+\varphi_{j+1};$$

$$a_{i,k}=a_i(r_{k-1}+jh_k);\qquad \varphi_{k,j}=\varphi(r_{k-1}+jh_k).$$

If we assume that it is always possible to choose a partition of the intervals $[r_{k-1},r_k]$ into $m_k$ parts such that

$$h_k=h_{k+1}=h\quad (k=1,2,\dots,s), \tag{8}$$

then equations (5), under conditions (6) and (7), form a system of linear homogeneous equations of order $n\sum_{k=1}^{s}(m_k-1)$ with a symmetric matrix.

Let $\lambda$ and $\lambda^{(h)}$ be the first, i.e., the smallest in absolute value, eigenvalues of problems (1)–(4) and (5)–(7), respectively.

**Theorem *.** *If, for all $i,j,k$ satisfying the inequalities $1\le i,j\le n,\ 1\le k\le s$, the following conditions hold: 1) $a_i$ are constant on $[r_{k-1},r_k]$; 2) $b_{i,j},c_{i,j}$ are twice continuously differentiable on $[r_{k-1},r_k]$; 3) $a_i>0$; 4) $b_{i,j}=b_{j,i},\ c_{i,j}=c_{j,i}$ for all $x$ from $(r_{k-1},r_k)$; 5) for any numbers $\xi_1,\xi_2,\dots,\xi_n$ and all $x$ from $(r_{k-1},r_k)$ one has*

$$\sum_{i,j=1}^{n} c_{i,j}\xi_i\xi_j\ge\alpha\sum_{i=1}^{n}\xi_i^2,\qquad \alpha=\text{const}>0,$$

*and also conditions (8), then as $h\to 0$*

$$|\lambda - \lambda^{(h)}| < ch^2, \tag{9}$$

*where $c$ is a constant independent of $h$.*

To prove the theorem, first of all the inequalities are established

$$\lambda \leq \frac{D[u,u]}{I[u,u]}, \qquad \lambda^{(h)} \leq \frac{D_h[u^{(h)}, u^{(h)}]}{I_h[u^{(h)}, u^{(h)}]} \tag{10}$$

for arbitrary functions $u$ and $u^{(h)}$ satisfying conditions (2)–(4) and (6)–(7), respectively. Here we have put

$$D[u,v] = \sum_{i=1}^{n} \sum_{k=1}^{s} \int_{r_{k-1}}^{r_k} a_i \frac{du_i}{dx} \frac{dv_i}{dx}\, dx - \sum_{i,l=1}^{n} \sum_{k=1}^{s} \int_{r_{k-1}}^{r_k} b_{i,l} u_i v_l \, dx +$$

$$+ \sum_{i=1}^{n} \sum_{k=1}^{s-1} \frac{a_i(r_k+0) - a_i(r_k-0)}{r_k} u_i(r_k) v_i(r_k),$$

$$D_h[u^{(h)}, v^{(h)}] = \sum_{i=1}^{n} \sum_{k=1}^{s} a_{i,k} h \sum_{j=0}^{m_k-1} \frac{\Delta u^{(h)}_{i,h,j}}{h} \frac{\Delta v^{(h)}_{i,h,j}}{h}$$

$$- \sum_{i,l=1}^{n} \sum_{k=1}^{s} \sum_{j=0}^{m_k-1} b_{i,l,k,j} u^{(h)}_{i,k,j} v^{(h)}_{l,k,j} + \sum_{i=1}^{n} \sum_{k=1}^{s-1} \frac{a_{i,k+1} - a_{i,k}}{r_k} u^{(h)}_{i,h,m_k} v^{(h)}_{i,h,m_k},$$

$$I[u,v] = \sum_{i,l=1}^{n} \sum_{k=1}^{s} \int_{r_{k-1}}^{r_k} c_{i,l} u_i v_l \, dx;$$

$$I_h[u^{(h)}, v^{(h)}] = \sum_{i,l=1}^{n} \sum_{k=1}^{s} h \sum_{j=0}^{m_k-1} c_{i,l,k,j} u^{(h)}_{i,h,j} v^{(h)}_{l,h,j}.$$

Equality in (10) occurs only in the cases when $u$ and $u^{(h)}$ are the first eigenfunctions of problems (1)–(4) and (5)–(7), respectively.

Further, starting from the first eigenfunction $u$ of problem (1)–(4), which is defined, in particular, at the nodes $r_{k-1} + jh$ $(k = 1, 2, \ldots, s;$

* This theorem is analogous to the corresponding theorem of Collatz (1) for the case of a single equation ($n = 1$) and absence of matching ($s = 1$).

$j = 0, 1, \dots, m_k - 1$), the difference is estimated as

$$\lambda^{(h)} - \lambda \leq \frac{D_h[u, u]}{I_h[u, u]} - \frac{D[u, u]}{I[u, u]} + O(h^2) < c_1 h^2. \tag{11}$$

Similarly, proceeding from the first difference eigenfunction $u^{(h)}$ of problem (5) –(7), extended linearly on $(r_{k-1} + jh,\ r_{k-1} + (j+1)h)$ $(j = 0, 1, \dots, m_k - 1;\ k = 1, 2, \dots, s;\ i = 1, 2, \dots, n)$:

$$\bar{u}_i(x) = u^{(h)}_{i,k,j} + \frac{\Delta u^{(h)}_{i,k,j}}{h}\,[x - (r_{k-1} + jh)],$$

we establish the estimate

$$\lambda - \lambda^{(h)} \leq \frac{D[\bar{u}, \bar{u}]}{I[\bar{u}, \bar{u}]} - \frac{D_h[u^{(h)}, u^{(h)}]}{I_h[u^{(h)}, u^{(h)}]} < c_2 h^2. \tag{12}$$

Since $c_1$ and $c_2$ do not depend on $h$ (to prove uniform boundedness in $h$ of $c_2$, it was necessary, in particular, to use Courant' s inequality ($^2$), § 12), the theorem follows immediately from (11) and (12).

We note that, although the first derivatives in the matching conditions (4) are approximated in the crudest way, namely with accuracy $O(h)$, the final error (see (9)) in the solution is majorized by a quantity of order $O(h^2)$. This circumstance does not occur, for example, in the Neumann problem: when the Laplace equation is approximated at interior nodes with accuracy $O(h^2)$, and the normal derivative at boundary nodes with accuracy $O(h)$, the error in the solution is determined by the cruder approximation and is $O(h)$ ($^3$).

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## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*

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