ON FILTRATION IN A RECTANGULAR COFFERDAM WITH A VERY LARGE HEIGHT OF CAPILLARY RISE

`![Fig. 1 and Fig. 2](image)`

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Fig. 1 and Fig. 2

Figure 1: Fig. 1 and Fig. 2

Abstract

Full Text

Hydraulics

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ON FILTRATION IN A RECTANGULAR COFFERDAM WITH A VERY LARGE HEIGHT OF CAPILLARY RISE

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Hydrodynamic schemes of filtration taking into account a fully saturated capillary zone were first broadly considered by V. V. Vedernikov ([1]-[3]), who at the same time gave solutions for several of the simplest cases of filtration from canals. Subsequently, the range of problems considered in this formulation was somewhat extended (-); however, filtration schemes for dams taking capillarity into account were not specially investigated. Such an attempt was undertaken, in a rigorous formulation, by the author ( ), and, in a more approximate one, by Don-Kerkhem ( ). One should also note the experiments of A. Rousso-Spena ([1]), close in their formulation to these theoretical investigations, and the experiments on the investigation of filtration in a rectangular cofferdam with incomplete saturation of the capillary zone, carried out by Lutin and Dje ([11]).

Fig. 1 **Fig. 2**

In the present article a solution is given for the case of filtration in a very high (infinitely high) rectangular cofferdam with a very large (infinite) height of capillary rise. The scheme of the motion is shown in Fig. 1. The capillary zone is assumed to be fully saturated, and the motion in it to obey Darcy' s law.

Introduce the complex potential $f = \varphi + i\psi$, where

$$\varphi = -\varkappa(p/\gamma + y) + C,$$

where $\varkappa$ is the coefficient of filtration, and the complex coordinate of the region of motion $z = x + iy$. Let along the impermeable base AF we have $\psi = 0$. Then the line BCD will be the streamline $\psi = Q$, where Q is the filtration discharge per unit cross section of the cofferdam. The vertical segments AB and EF are equipotentials $\varphi = -\varkappa H$ and $\varphi = -\varkappa h_0$, respectively. The vertical seepage face DE is an isobar.

The region of the complex filtration velocity $\omega = u - iv = df/dz$ has the form shown in Fig. 2, since, by virtue of the constancy of pressure along the seepage face, we have on the line DE

$$v = -\varkappa \quad (^{12}).$$

Introduce an auxiliary complex variable ζ, onto the lower half-plane of which we shall map the region of motion under the following ···

correspondence of points: $\zeta(A) = +1,\ \zeta(B) = +b,\ \zeta(C) = \infty,\ \zeta(D) = -a,\ \zeta(E) = -c,\ \zeta(F) = -1$. Then

$$\frac{dz}{d\zeta} = -\frac{L}{\pi}\frac{1}{\sqrt{1-\zeta^2}}, \qquad z = -\frac{L}{\pi}\arcsin\zeta, \tag{1}$$

$$H = \frac{L}{\pi}\operatorname{arch} b, \qquad h = \frac{L}{\pi}\operatorname{arch} a, \qquad h_0 = \frac{L}{\pi}\operatorname{arch} c. \tag{2}$$

It is evident that $b > a > c$, and the quantity a is not known in advance and must be found from the solution.

Using the Christoffel-Schwarz formula, map the domain w onto ζ:

$$W = iN\int_{-\infty}^{\zeta}\frac{(\mu-\zeta)\,d\zeta}{(-c-\zeta)(b-\zeta)\sqrt{(b-\zeta)(-a-\zeta)}}. \tag{3}$$

Here μ is the image of the point having the minimum horizontal velocity on the segment $EFAB$.

Taking into account that $w(-a) = i\varkappa$ and that $v(-c-0) - v(-c+0) = -\varkappa$, we find

$$\text{on } EFAB: \quad w = \frac{\varkappa}{\pi}\left[\operatorname{arch}\left(\frac{p+q\zeta}{c+\zeta}\right) + \sqrt{\frac{a+\zeta}{b-\zeta}}\arccos q\right], \tag{4'}$$

$$\text{on } BCD: \quad w = i\frac{\varkappa}{\pi}\left[\arccos\left(\frac{p+q\zeta}{c+\zeta}\right) - \sqrt{\frac{-a-\zeta}{b-\zeta}}\arccos q\right], \tag{4''}$$

$$\text{on } DE: \quad w = i\varkappa + \frac{\varkappa}{\pi}\left[\operatorname{arch}\left(-\frac{p+q\zeta}{c+\zeta}\right) + \sqrt{\frac{a+\zeta}{b-\zeta}}\arccos q\right]. \tag{4'''}$$

Here the following notation has been introduced:

$$\frac{2ba-(b-a)c}{b+a}=p, \qquad \frac{b+2c-a}{b+a}=q. \tag{5}$$

Since $w = df/dz$, we have

$$f=\int w(\zeta)z'(\zeta)\,d\zeta. \tag{6}$$

To determine the magnitude of the seepage-face segment, one uses the equation relating the pressures at points of the upper and lower pools, for example A and F:

$$\varkappa(H-h_0)=f(F)-f(A)=-\int_{-1}^{+1} w(\zeta)z'(\zeta)\,d\zeta. \tag{7}$$

In the case of a difference Δp between the atmospheric pressures above the pools, the quantity $\varkappa\Delta p/\gamma$ should be added to the left-hand side of the last relation; however, this case is of no practical interest.

Expanding (7) according to (1) and (4), we obtain

$$H-h_0=\frac{L}{\pi^2}\left[\int_{-1}^{+1}\operatorname{arch}\left(\frac{p+q\zeta}{c+\zeta}\right)\frac{d\zeta}{\sqrt{1-\zeta^2}}+\arccos q\int_{-1}^{+1}\frac{\sqrt{a+\zeta}\,d\zeta}{\sqrt{b-\zeta}\sqrt{1-\zeta^2}}\right]. \tag{8}$$

Making in the first integral the substitution $\zeta=\sin\theta$ and taking (2) into account, we obtain, for determining the parameter a, the equation

$$\pi(\operatorname{arch} b-\operatorname{arch} c)=\int_{-\pi/2}^{+\pi/2}\operatorname{arch}\left(\frac{p+q\sin\theta}{c+\sin\theta}\right)d\theta+\arccos q\int_{-1}^{+1}\frac{\sqrt{a+\zeta}\,d\zeta}{\sqrt{b-\zeta}\sqrt{1-\zeta^2}}. \tag{9}$$

The second of the integrals entering here can be expressed in terms of complete elliptic integrals of the first and third kinds, or in terms of elliptic integrals of the first and second kinds, with modulus

$$k=\sqrt{\frac{2}{a+1}\,\frac{b+a}{b+1}} \tag{10}$$

as follows:

$$\int_{-1}^{+1}\frac{\sqrt{a+\zeta}\,d\zeta}{\sqrt{b-\zeta}\sqrt{1-\zeta^2}}=2\sqrt{\frac{a+1}{b+1}}\left\{K-\frac{b-1}{a+1}[\Pi(n)-K]\right\}=$$

$$= 2\sqrt{\frac{a+1}{b+1}}\,K - 2\left[KE(\omega) - E\Big(\frac{\pi}{2}\Big)\,F(\omega)\right]. \qquad (11)$$

Here

$$\Pi(n) = \int_0^{\pi/2} \frac{d\theta}{(1 + n\sin^2\theta)\sqrt{1 - k^2\sin^2\theta}},$$

$$\omega = \arcsin\sqrt{\frac{a+1}{b+a}}, \qquad n = -\frac{2}{b+1}.$$

From equation (9) it is easy, in particular, to obtain the asymptotic dependence of h/H on L/H for small values of L/H:

$$\frac{h}{H} = 1 - \lambda\frac{L}{H} \qquad \Big(\lambda = -\frac{\ln s}{\pi} \simeq 0.50446\Big). \qquad (12)$$

The constant s here is the root of the equation

$$\sqrt{s}\,\arccos\frac{1-s}{1+s} = \ln\frac{1+s}{4s}. \qquad (13)$$

To determine the magnitude of the filtration discharge one may use the condition

$$iQ = f(B) - f(A) = \int_1^b w(\zeta)z'(\zeta)\,d\zeta, \qquad (14)$$

expanding which, we obtain

$$Q = \frac{\chi L}{\pi^2}\left[\int_1^b \operatorname{ar\,ch}\Big(\frac{p+q\zeta}{c+\zeta}\Big)\frac{d\zeta}{\sqrt{\zeta^2-1}} + \arccos q\int_1^b \frac{\sqrt{a+\zeta}\,d\zeta}{\sqrt{b-\zeta}\sqrt{\zeta^2-1}}\right] \qquad (15)$$

or

$$Q = \frac{\chi L}{\pi^2}\left[\int_0^{\operatorname{ar\,ch} b} \operatorname{ar\,ch}\Big(\frac{p+q\,\operatorname{ch} t}{c+\operatorname{ch} t}\Big)\,dt + \arccos q\int_1^b \frac{\sqrt{a+\zeta}\,d\zeta}{\sqrt{b-\zeta}\sqrt{\zeta^2-1}}\right]. \qquad (16)$$

The second integral here too can be expressed in terms of elliptic integrals of the first and second kinds, or in terms of complete elliptic integrals of the first and third kinds,

Fig. 3

Figure 2: Fig. 3

$$\int_1^b \frac{\sqrt{a+\zeta}\,d\zeta}{\sqrt{b-\zeta}\sqrt{\zeta^2-1}} = 2\sqrt{\frac{a+1}{b+1}}\left\{K + \frac{2}{a+1}[\Pi(n) - K]\right\} =$$

$$= 2\sqrt{\frac{a+1}{b+1}}\,K + 2\left[\frac{\pi}{2} - EF'(\omega) - KE'(\omega) + KF'(\omega)\right]. \qquad (17)$$

Here

$$\omega = \arcsin\sqrt{\frac{a+1}{b+a}}, \qquad n = -\frac{b-1}{b+1},$$

and the modulus of the elliptic integrals is

$$k = \sqrt{\frac{a-1}{a+1}\frac{b-1}{b+1}}. \qquad (18)$$

Numerical calculations were carried out using the above formulas for the case $h_0 = 0$ ($c = 1$). The results of the calculations are presented in Fig. 3, where, as functions of L/H, the dependences h/H (1), $Q/\chi H$ (3), as well as the dependences h^0/H (2) and $Q^0/\chi H$ (4), obtained from the rigorous solution of the problem in the absence of soil capillarity, are plotted. Curve 2, obtained from the solution of P. Ya. Polubarinova-Kochina ([13,14]), is taken from the dissertation ([15]), and curve 4 is constructed by the Dupuit–Charny formula ([16]).

Fig. 3

According to the solution of B. K. Riesenkampf ([17]), which can be interpreted in the sense of inflow from infinity to an inclined slope at an infinitely large height of capillary rise, for inflow to a vertical slope the ratio $Q/\chi H = 1$. The results of the numerical calculations show that already for $L/H > 2$ we have $Q/\chi H \approx 1$. This once again confirms the possibility of a local consideration of separate regions of the filtration flow without taking into account the conditions on sufficiently remote parts of the boundary of the region of motion ([15]).

It should be noted that in the hydraulic formulation ([18,10]), with unlimited growth of the height of capillary rise, the magnitude of the filtration discharge Q also grows without bound, which obviously does not correspond to reality. This requires the establishment of criteria for the applicability of the corresponding hydraulic formulas.

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Note: Figure translations are in progress. See original paper for figures.