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Abstract

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MATHEMATICS

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EMBEDDING THEOREMS FOR A SPACE WITH A METRIC DEGENERATING AT A FINITE NUMBER OF INTERIOR POINTS OF A BOUNDED DOMAIN

(Presented by Academician S. L. Sobolev, 16 XI 1956)

Let D' be a finite domain situated in the plane of the variables (x, y). Denote by Γ' the closed curve bounding the domain D', and suppose that Sobolev' s embedding theorems ([1]) hold for it. For simplicity of exposition we shall assume that the origin of coordinates lies inside the domain D' and that the complete boundary is $\Gamma = \Gamma' + (0, 0)$.

1. Let $\bar{\Omega}^0$ be the manifold of all functions continuous in D', having bounded piecewise-continuous first derivatives and vanishing in some boundary strip of the domain and in some neighborhood of the point $(0, 0)$. The strip and the neighborhood are taken separately for each function. Denote by Gu^0 the gradient of the function $u^0 \in \bar{\Omega}^0$:

$$Gu^0 = \left(\frac{\partial u^0}{\partial x}, \frac{\partial u^0}{\partial y}\right).$$

The manifold composed of the elements Gu^0 will be denoted by $\bar{R}^0$. In $\bar{R}^0$ we introduce a scalar product by the formula

$$\{Gu^0, Gv^0\} = \iint_{D'} \left[b_{11}\frac{\partial u^0}{\partial x}\frac{\partial v^0}{\partial x} + b_{12}\frac{\partial u^0}{\partial x}\frac{\partial v^0}{\partial y} + b_{12}\frac{\partial u^0}{\partial y}\frac{\partial v^0}{\partial x} + b_{22}\frac{\partial u^0}{\partial y}\frac{\partial v^0}{\partial y}\right] dx\, dy, \tag{1}$$

where the following restrictions are imposed on the coefficients b_{11}, b_{12}, and b_{22}:

1) b_{11}, b_{12}, and b_{22} are continuous in $D^\delta = D' \cap (r \geq \delta)$, where $\delta > 0$ is arbitrary, $r = \sqrt{x^2 + y^2}$;
2) either b_{11} and b_{22} tend to infinity at the point $(0, 0)$, or one of them tends to zero at the point $(0, 0)$;

3) for any real numbers ξ_1 and ξ_2 such that $\xi_1^2 + \xi_2^2 > 0$, the quadratic form

$$B_1(\xi_1, \xi_2; x, y) \equiv b_{11}\xi_1^2 + 2b_{12}\xi_1\xi_2 + b_{22}\xi_2^2 \geq 0 \tag{2}$$

everywhere in the domain $\bar{D}' = D' \cup \Gamma$, and the equality sign is attained only at the point $(0, 0)$ and in the case when either b_{11} or b_{22} tends to zero at this point. Denote by $\dot{R}$ the closure of the space $\bar{R}^0$ in the metric (1).

It is easy to see that any element $g \in \dot{R}$ is equal to a system of generalized first partial derivatives of a function u in the domain D', i.e.

$$g = Gu = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right).$$

For $Gu, Gv \in \dot{R}$, the scalar product $\{Gu, Gv\}$ can also be computed by formula (1), where the integral must be understood in the Lebesgue sense. Denote by $\dot{\Omega}$ the manifold of functions u obtained as a result of the closure process indicated above. Obviously,

$$G\dot{\Omega} = \dot{R}.$$

Theorem 1. 1) On the part of the boundary Γ', every function from $\dot{\Omega}$ has mean value zero.

2) If, for $\alpha_i \geq 0,\ i = 1, 2$, the conditions

$$c^2 r^{\alpha_1} \leq b_{11} \leq C^2 r^{\alpha_1}, \qquad c_1^2 r^{\alpha_2} \leq b_{22} \leq C_1^2 r^{\alpha_2}, \tag{3}$$

are fulfilled, then any function $f(x, y)$ having bounded piecewise-continuous first derivatives in D' and vanishing on Γ', with

a) $\{Gf, Gf\} < +\infty$;

b) $|f| \leq c^2 r^{-\bar{\alpha}/2}$, where $\bar{\alpha} = \min(\alpha_1, \alpha_2) > 0$; $|f| \leq c^2 |\ln r|^{1/2}$, where $\bar{\alpha} = 0$,

belongs to $\dot{\Omega}$.

3) If, for arbitrary ξ_1 and ξ_2,

$$(\xi_1^2 + \xi_2^2) r^{\alpha} \leq c^2 B_1(\xi_1, \xi_2; x, y) \tag{4}$$

for $\alpha < 0$, then every function from $\dot{\Omega}$ vanishes at the point $(0, 0)$.

4) If, for $\alpha \neq 0$, inequality (4) is satisfied, then for any function $u \in \dot{\Omega}$ the estimate

$$\iint_{D'} \sigma(x, y) u^2(x, y)\, dx\, dy \leq c^2 \{Gu, Gu\},$$

holds, where $\sigma(x, y) > 0$ for $x^2 + y^2 > 0$ and is a sufficiently smooth function, with

$$\sigma(x, y) = O(r^{\alpha-2} |\ln r|^{-1-\varepsilon_0}), \qquad \varepsilon_0 > 0.$$

We note that σ may, for example, be taken equal to the expression under the O-sign.

It follows from this theorem that, when conditions (3) are satisfied for $\alpha_i \geq 0$, the space $\dot{\Omega}$ contains functions vanishing on Γ'; at the point $(0, 0)$, functions from $\dot{\Omega}$ may take arbitrary values. If, for $\alpha_i < 0$, condition (4) is satisfied, then all functions from $\dot{\Omega}$ vanish on $\Gamma = \Gamma' + (0, 0)$. If, for $\alpha \neq 0$ and $\alpha < 2$, condition (4) is satisfied, then all functions from $\dot{\Omega}$ are square-summable over the domain D'.

2. Let Ω_1^0 be the manifold of all continuous functions in D' having bounded piecewise-continuous second derivatives and vanishing in some boundary strip of the domain D' and in some neighborhood of the point $(0, 0)$. The strip and the neighborhood are chosen individually for each function. On the set of functions $u^0 \in \Omega_1^0$, define an operator of gradient type:

$$Gu^0 = \left(\frac{\partial^2 u^0}{\partial x^2}, \frac{\partial^2 u^0}{\partial x\, \partial y}, \frac{\partial^2 u^0}{\partial y^2} \right).$$

As in remark (3), introduce the space $\dot{\Omega}_1$ and the space $\dot{R}_1$ with metric given by the scalar product

$$\begin{aligned} \{Gu, Gv\} = \iint_{D'} \Bigg[& a_{1111} \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial x^2} + a_{1212} \frac{\partial^2 u}{\partial x\, \partial y} \frac{\partial^2 v}{\partial x\, \partial y} + a_{2222} \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 v}{\partial y^2} \\ & + \frac{1}{2} a_{1112} \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial x\, \partial y} + \frac{1}{2} a_{1112} \frac{\partial^2 u}{\partial x\, \partial y} \frac{\partial^2 v}{\partial x^2} + \frac{1}{2} a_{1222} \frac{\partial^2 u}{\partial x\, \partial y} \frac{\partial^2 v}{\partial y^2} + \frac{1}{2} a_{1222} \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 v}{\partial x\, \partial y} \\ & + \frac{1}{2} a_{1122} \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial y^2} + \frac{1}{2} a_{1122} \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 v}{\partial x^2} \Bigg] \, dx\, dy, \end{aligned} \tag{5}$$

where the following restrictions are imposed on the coefficients of the integrand:

1) they are all continuous in the domain D'_δ, where $\delta > 0$ is arbitrary;

2) either $a_{1111}, a_{1212}, a_{2222}$ tend to infinity at the point $(0, 0)$, or one of them tends to zero at the point $(0, 0)$;

3) for any real numbers $\xi_{11}, \xi_{12}, \xi_{22}$ such that $\xi_{11}^2 + \xi_{12}^2 + \xi_{22}^2 > 0$, the quadratic form

$$B(\xi_{11}, \xi_{12}, \xi_{22}; x, y) \equiv a_{1111}\xi_{11}^2 + a_{1212}\xi_{12}^2 + a_{2222}\xi_{22}^2 + a_{1112}\xi_{11}\xi_{12} + a_{1222}\xi_{12}\xi_{22} + a_{1122}\xi_{11}\xi_{22} \geq 0$$

everywhere in the domain $\overline{D'} = D' \cup \Gamma$, and the equality sign can occur only at the point $(0, 0)$. It is easy to see that any element $g \in \dot{R}_1$ is an operator of gradient type, i.e.,

$$g = Gu = \left(\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y} \right),$$

where the derivatives are understood in the generalized sense, and $G\dot{\Omega}_1 = \dot{R}_1$.

Sometimes we shall assume that

$$c_1^2 r^{\alpha_1} \leq a_{1111} \leq C_1^2 r^{\alpha_1}, \tag{6}$$

$$c_2^2 r^{\alpha_2} \leq a_{1212} \leq C_2^2 r^{\alpha_2}, \tag{7}$$

$$c_3^2 r^{\alpha_3} \leq a_{2222} \leq C_3^2 r^{\alpha_3}, \tag{8}$$

$$0 \leq r^{\alpha_1}\xi_{11}^2 \leq c^2 B(\xi_{11}, \xi_{12}, \xi_{22}; x, y), \tag{9}$$

$$0 \leq r^{\alpha_2}\xi_{12}^2 \leq c^2 B(\xi_{11}, \xi_{12}, \xi_{22}; x, y), \tag{10}$$

$$0 \leq r^{\alpha_3}\xi_{22}^2 \leq c^2 B(\xi_{11}, \xi_{12}, \xi_{22}; x, y). \tag{11}$$

Theorem 2. 1) On the part of the boundary Γ', every function from $\dot{\Omega}_1$ has mean value zero together with its first derivatives.

2) If, for $0 \leq \alpha < 2$, for any real $\xi_{11}, \xi_{12}, \xi_{22}$, the condition

$$(\xi_{11}^2 + \xi_{12}^2 + \xi_{22}^2)\, r^\alpha \leq c^2 B(\xi_{11}, \xi_{12}, \xi_{22}; x, y), \tag{12}$$

is satisfied, then every function $u \in \dot{\Omega}_1$ vanishes at the point $(0, 0)$.

3) If for $\alpha < 0$ condition (12) is satisfied, then every function $u \in \dot{\Omega}_1$ vanishes at the point $(0, 0)$ together with its first-order derivatives.

4) If conditions (6), (7), (8), (9), (10), and (11) are satisfied for $0 \leq \alpha_i$, then every function $f(x, y)$ which has bounded piecewise-continuous second derivatives in D'_δ and vanishes on Γ' together with its first derivatives, and such that

a) $\{Gf, Gf\} < +\infty$;

b) $|f| \le c_1^2 r^{(2-\bar\alpha)/2}$ for $\bar\alpha \ne 2$, $|f| \le c_1^2 |\ln r|^{1/2}$ for $\bar\alpha = 2$;

c) $\left|\dfrac{\partial f}{\partial x}\right| \le c_2^2 |\ln r|^{1/2}$, $\left|\dfrac{\partial f}{\partial y}\right| \le c_3^2 |\ln r|^{1/2}$ for $\bar\alpha = 0$;

d) $\left|\dfrac{\partial f}{\partial x}\right| \le c_2^2 r^{-\bar\alpha/2}$, $\left|\dfrac{\partial f}{\partial y}\right| \le c_3^2 r^{-\bar\alpha/2}$ for $\bar\alpha > 0$, where $\bar\alpha = \min(\alpha_1, \alpha_2, \alpha_3)$,

belongs to $\dot\Omega_1$.

5) If, for $\alpha \ne 0$, condition (12) is satisfied, then for functions $u \in \dot\Omega_1$ the estimates

$$\iint_{D'} \sigma_0(x,y) u^2(x,y)\, dx\, dy \le c^2\{Gu, Gu\},$$

$$\iint_{D'} \sigma_1(x,y) \left(\frac{\partial u}{\partial x}\right)^2 dx\, dy \le c_1^2\{Gu, Gu\},$$

$$\iint_{D'} \sigma_1(x,y) \left(\frac{\partial u}{\partial y}\right)^2 dx\, dy \le c_2^2\{Gu, Gu\},$$

where c^2, c_1^2, and c_2^2 do not depend on the function u; σ_0 and σ_1 are sufficiently smooth functions, $\sigma_0 > 0$ and $\sigma_1 > 0$ for $x^2 + y^2 > 0$, and

$$\sigma_0(x,y) = \begin{cases} O\left(r^{\alpha-4} |\ln r|^{-1-\varepsilon_0}\right), & \text{for } \alpha \ne 2, \\ O\left(r^{-2} |\ln r|^{-2-\varepsilon_0}\right), & \text{for } \alpha = 2; \end{cases}$$

$$\sigma_1(x,y) = O\left(r^{\alpha-2} |\ln r|^{-1-\varepsilon_0}\right), \qquad \varepsilon_0 > 0 \text{ arbitrary.}$$

Assertion 1 follows from the equivalence of the metrics (5) and $W_2^{(2)}$, D'_δ, and from the embedding theorems of S. L. Sobolev ([1]).

To prove assertion 2, let us extend the functions $u \in \dot\Omega_1$ by zero to the domain D'. We shall say that $u \in W_2^{(2-\alpha/2)}(D')$ if u has generalized first-order derivatives u_x and u_y satisfying the conditions

$$\left(\iint_{D'} |u_x(x+\Delta x, y+\Delta y) - u_x(x,y)|^2\, dx\, dy\right)^{1/2} \le c^2(\Delta x^2 + \Delta y^2)^{(2-\alpha)/4},$$

$$\left(\iint_{D'} |u_y(x+\Delta x, y+\Delta y) - u_y(x,y)|^2\, dx\, dy\right)^{1/2} \le c^2(\Delta x^2 + \Delta y^2)^{(2-\alpha)/4},$$

where c^2 does not depend on Δx and Δy. It is shown that if $u \in \dot{\Omega}_1$, then $u \in W_2^{(2-\alpha/2)}(D')$. It follows that the function u will be continuous in the domain D' ([2],[4]), and consequently assertion 2 of the theorem is proved.

To prove assertion 4, introduce the function

$$\chi_\delta(x,y) = \begin{cases} 0, & 0 \le r < \delta, \\ \{1 - [(\ln|\ln r|)^\varepsilon - (\ln|\ln \delta_1|)^\varepsilon]^2\}^2, & \delta \le r \le \delta_1, \\ 1, & r > \delta_1, \end{cases}$$

where $(\ln|\ln \delta|)^\varepsilon - (\ln|\ln \delta_1|)^\varepsilon = 1$, $0 < \varepsilon < 1/2$. Obviously, the function $f_\delta = f\chi_\delta \in \dot{\Omega}_1$. It is shown that Gf_δ converges in the metric (5) to Gf. Hence it follows that $f \in \dot{\Omega}_1$.

Assertions 3 and 5 are proved by means of ordinary estimates.

On the basis of this theorem one can draw the following conclusions.

If condition (12) is satisfied for $\alpha < 0$, then every function from $\dot{\Omega}_1$ vanishes, together with its first derivatives, on the entire boundary Γ.

If condition (12) is satisfied for $0 \le \alpha < 2$, then every function from $\dot{\Omega}_1$ vanishes, together with its first derivatives, on Γ'; at the point $(0,0)$ only the function itself vanishes.

If conditions (6), (7), (8), (9), (10), and (11) are satisfied for $\alpha_i \ge 0$, then $\dot{\Omega}_1$ contains functions that vanish, together with their first derivatives, on Γ'; at the point $(0,0)$ both the function and its first derivatives may take arbitrary values.

If condition (12) is satisfied, then for $\alpha < 4$ all functions from $\dot{\Omega}_1$ are square-summable over the domain D'; for $\alpha < 2$ their first derivatives are square-summable over the domain D'.

All the results of the second part of the note are easily generalized to the case when derivatives of order m enter the scalar product.

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Note: Figure translations are in progress. See original paper for figures.