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Abstract

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ON THE QUESTION OF PERIODIC SOLUTIONS OF NONLINEAR SYSTEMS WITH A SMALL PARAMETER

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In the theory of nonlinear oscillations it is of interest to obtain general qualitative theorems concerning the behavior of periodic solutions under small constantly acting perturbations, independently of any special properties of these perturbations. In particular, one may pose the following question: if a periodic solution is asymptotically stable (i.e., if it is preserved in some manner under arbitrary small changes of the initial data), can one assert that under small periodic constantly acting perturbations there exists a periodic solution in some neighborhood of it? Obtaining an answer to this question is the aim of the present note. Partial results in this direction were obtained by Kh. Antosiewicz ([1]) and A. Halanay ([2]).

We shall consider the system of equations

$$\frac{dx}{dt} = X(x,t) + \mu Y(x,t,\mu), \tag{1}$$

where $x = (x_1, \dots, x_n)$, $X = (X_1, \dots, X_n)$, $Y = (Y_1, \dots, Y_n)$, and:

1) $X(x,t)$, $Y(x,t,\mu)$ are continuous vector-functions, periodic in t with period ω, defined for $|x| \le \rho$ and all t $(|x| = \sqrt{x_1^2 + \dots + x_n^2})$;

2) in any cylinder $|x| \le \rho$, $0 \le t \le T$, uniqueness of solutions and their continuous dependence on the initial values and on the parameter μ are ensured;

3) the trivial solution $x = 0$ of the generating system

$$\frac{dx}{dt} = X(x,t) \tag{2}$$

is asymptotically stable (we note that the general case of a nonzero asymptotically stable solution is easily reduced to this case by a simple change of the unknown function).

Theorem. *If conditions 1)–3) are fulfilled, then the system (1), for $|\mu| < \mu_0$, admits periodic solutions situated in a neighborhood * of the solution $x = 0$ of the system (2).*

Proof. According to the results of ([3]), the system (2) possesses a Lyapunov function periodic in t, which is **: a) positive definite and b) strictly decreasing along integral curves as t increases.

It is easy to verify that, for sufficiently small $C > 0$, the component of the set $\{x \mid V(x,0) \le C\}$ to which the origin belongs is located–

* In the sense that as $\mu \to 0$ these solutions tend to $x = 0$.

** From the proof given in ([3]), it is easy to see that, for the fulfillment of conditions a) and b), the existence of partial derivatives of the right-hand side of the system (2) is immaterial.

is placed in any prescribed spherical neighborhood of the origin. From the periodicity of $V(x,t)$ it follows that

$$\{x \mid V(x,0) \le C\} = \{x \mid V(x,\omega) \le C\}.$$

Let $x(x_0, 0; t)$ be a solution of system (2) passing at the time $t = 0$ through x_0. The mapping φ:

$$\varphi(x_0) = x(x_0, 0; \omega) \tag{3}$$

is continuous (indeed, homeomorphic).

Let D be the component of the set $\{x \mid V(x,0) < C\}$ containing the origin, and let $F = \overline{D} \setminus D$. On the set F, $V(x,0) = V(x,\omega) = C$. Then

$$\varphi(\overline{D}) \subset D \tag{4}$$

(the image of the set $\overline{D}$ does not intersect the set F, since along the integral curves the Lyapunov function strictly decreases). Since $\varphi(\overline{D})$ is compact and connected, it is easy to find a connected polyhedron P such that

$$\varphi(\overline{D}) \subset \operatorname{int} P \subset D \subset \overline{D} \tag{5}$$

($\operatorname{int} P$ is the set of interior points of P). The polyhedron P may, for example, be constructed from sufficiently small cubes with edges parallel to the coordinate axes.

From (5) it follows that

$$\varphi(P) \subset \operatorname{int} P. \tag{6}$$

Topological fixed-point lemma. *If a continuous mapping φ of a polyhedron P into itself, iterated k times, gives a mapping φ^k, $\varphi^k(x) = \varphi(\varphi^{k-1}(x))$, homotopic (or even merely homologous) to a constant mapping, then the Lefschetz number of the mapping φ is equal to one*

$$\Lambda(\varphi) = 1, \tag{7}$$

and, consequently, φ has a fixed point.

The proof is obtained from the following algebraic observation: the trace of a nilpotent endomorphism of a free Abelian group with m generators is always equal to zero.

Indeed, in a fixed basis the trace of an endomorphism is defined as the sum of the diagonal elements of the matrix of this endomorphism, also equal to the sum of the characteristic roots. But the characteristic roots of a nilpotent matrix are equal to zero.

By the definition of the Lefschetz number,

$$\Lambda(\varphi) = \sum_{r=0}^{n} (-1)^r \operatorname{Sp}(\psi_r), \tag{8}$$

where ψ_r is the endomorphism of the r-dimensional group of weak homologies of the polyhedron P, generated by the mapping φ:

$$\psi_r = \Delta_{00}^r \to \Delta_{00}^r \tag{9}$$

(the notation is taken from (4)). But for $r > 0$ the endomorphism ψ_r is nilpotent, and $\operatorname{Sp}(\psi_r) = 0$, since the endomorphism $(\psi_r)^k$ corresponds to the mapping φ^k, and φ^k, by assumption, is homologous to a constant mapping. In view of the connectedness of P, $\operatorname{Sp}(\psi_0) = 1$, and we obtain $\Lambda(\varphi) = 1$. The lemma is proved*.

* In Leray' s work (7) there is the following proposition: if, under a mapping φ of a compactum P into itself, the set $\bigcap_{k=1}^{\infty} \varphi^k(P)$ is a singleton, then $\Lambda(\varphi) = 1$ ((5), p. 179, theorem 20, corollary 20). It is easy to verify that the theorem of the present note can also be proved with the aid of this proposition.

Let us return to the proof of the theorem. Let $U_\varepsilon \subset P$ be a spherical neighborhood of the origin. From the asymptotic stability of the zero solution of system (2) it follows that there exists a k such that $\varphi^k(P) \subset U_\varepsilon$ (P is a polyhedron with property (6), and φ is the mapping from (4)). Indeed, for $k\omega > T_\varepsilon$,

$$\varphi^k(x_0) = x(x_0, 0; k\omega) \in U_\varepsilon.$$

Define for system (1) the mapping

$$\varphi_{(\mu)}(x_0) = x_{(\mu)}(x_0, 0; \omega), \tag{10}$$

where $x_{(\mu)}$ is a solution of system (1). By the continuous dependence of the solution on μ,

$$\varphi_{(\mu)}(P) \subset \operatorname{int} P, \qquad (\varphi_{(\mu)})^k(P) \subset U_\varepsilon \quad \text{for } |\mu| < \mu_0. \tag{11}$$

Then all the assumptions of the topological lemma are satisfied for the mapping $\varphi_{(\mu)}$. Indeed, the mapping $(\varphi_{(\mu)})^k$ is homotopic to a constant mapping, since the ball U_ε is contractible to a point. Consequently, $\varphi_{(\mu)}$ has a fixed point for $|\mu| < \mu_0$. Referring to the definition of this mapping, we conclude that there exists a point x_0 such that

$$\varphi_{(\mu)}(x_0) = x_0, \quad \text{i.e. } x_{(\mu)}(x_0, 0; t) = x_0, \tag{12}$$

and this means that there exists a periodic solution of system (1). Since the initial set $V(x, 0) < C$ could be taken in any neighborhood of the origin, it follows that as $\mu \to 0$ the periodic solution tends to the solution $x = 0$. The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.