ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A NONLINEAR BOUNDARY-VALUE PROBLEM IN BOUNDARY-LAYER THEORY

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Abstract

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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A NONLINEAR BOUNDARY-VALUE PROBLEM IN BOUNDARY-LAYER THEORY

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In boundary-layer theory the following boundary-value problem is well known:

$$\begin{aligned} &y'''(t) + 2y(t)y''(t) + 2\beta(k^2 - [y'(t)]^2) = 0 \quad (0 \le t < \infty), \\ &y(0) = y'(0) = 0, \qquad y'(\infty) = k \quad (\beta \ge 0,\ k > 0). \end{aligned} \tag{1}$$

This problem has been considered by a number of authors (see ([1]), references on pp. 80, 83, 113, 118, 129; see also the later works ([2–5])).

The purpose of this note is to investigate the asymptotic behavior, as $t \to \infty$, of the solution of the boundary-value problem (1).

It is of interest to obtain a method for investigating the asymptotic behavior of solutions of a broader class of boundary-value problems, including (1) as a special case.

Let $y(t)$ $(0 \le t < \infty)$ be a solution of the boundary-value problem (1).

Lemma 1.

$$y'(t) < k \qquad (0 \le t < \infty).$$

Proof. Suppose that the assertion is false and that τ is the first root of the equation $y'(t) = k$. Since $y'(t) < k$ $(0 \le t < \tau)$, we have $y''(\tau) \ge 0$. If $y''(\tau) = 0$, then, by the uniqueness theorem for the solution of a differential equation with prescribed initial conditions,

$$y(t) \equiv y(\tau) + (t - \tau)k,$$

i.e. $y'(0) = k$, which is impossible. Hence $y''(\tau) > 0$, i.e. $y'(t)$ increases at the point τ. But $y'(\infty) = k$, and therefore $y'(t)$ attains, at some point $t_1 > \tau$, its greatest value on the interval $[\tau, \infty)$. At the point t_1 we have $y'' = 0$, $y''' \le 0$; but $y'(t_1) > k$, whence

$$y'''(t_1) = -2\beta(k^2 - [y'(t_1)]^2) > 0$$

if $\beta > 0$. If, however, $\beta = 0$, then, as above,

$$y(t) \equiv y(t_1) + (t - t_1)y'(t_1),$$

i.e. $y'(0) = y'(t_1) > 0$.

The contradiction obtained in all cases proves Lemma 1.

We transform the boundary-value problem (1) to a form more convenient for further consideration. Let $t_0 \geq 0$ be the last root of the equation $y'(t) = 0$. Then for $t \geq t_0$ the function $y(t)$ increases, and therefore t, and hence also $y'(t)$, is a function of y. Making the substitution of the unknown function

$$v = k^2 - [y'(t)]^2 = v(y) \qquad (y_0 = y(t_0) \leq y < \infty),$$

we obtain the boundary-value problem

$$\begin{gathered} \sqrt{k^2 - v(y)}\, v''(y) + 2yv'(y) - 4\beta v(y) = 0 \quad (y_0 < y < \infty), \\ v(y_0) = k^2, \qquad v(\infty) = 0, \qquad 0 < v(y) < k^2 \quad (y_0 < y < \infty). \end{gathered} \tag{2}$$

Lemma 2.

$$v'(y) < 0 \qquad (y_0 < y < \infty).$$

Proof. At the roots of the equation $v'(y) = 0$

$$v''(y) = \frac{4\beta v}{\sqrt{k^2 - v}} > 0 \qquad (\text{for } \beta > 0),$$

i.e., all stationary points of v must be minima of $v(y)$, which is impossible; in the case $\beta = 0$, with $v'(y_1) = 0$ $(y_1 > y_0)$, we would have $v(y) \equiv v(y_1)$, which is also impossible. Lemma 2 is proved.

Lemma 3. *The function $v''(y)$ has, for $y > y_0$, at most one zero.*

Proof. From relations (2) it is clear that for $y > y_0$ the derivative $v'''(y)$ exists and

$$\sqrt{k^2 - v}\, v''' - \frac{1}{2\sqrt{k^2 - v}} v''v' + 2yv'' + 2(1 - 2\beta)v' = 0. \tag{3}$$

Hence, if $\beta \neq \frac{1}{2}$, then at the points where $v''(y) = 0$,

$$(2\beta - 1)v''' = \frac{2(2\beta-1)^2}{\sqrt{k^2 - v}}\, v' < 0$$

(by Lemma 2), i.e. v''' has one and the same sign. Our assertion follows from this.

Let now $\beta = \frac{1}{2}$. Then, if $v''(y_1) = 0$ $(y_1 > y_0)$, by the uniqueness theorem for a solution applied to equation (3),

$$v(y) \equiv v(y_1) + (y - y_1)v'(y_1) \qquad (y_0 < y < \infty).$$

Since $v(\infty) = 0$, we obtain $v'(y_1) = 0$, which contradicts Lemma 2. Lemma 3 is proved.

Corollary.

$$v'(\infty) = 0.$$

This relation, in view of the convergence of the integral $\int^{\infty} v'(y)\,dy$, follows at once from Lemma 3.

Lemma 4.

$$v''(\infty) = 0.$$

Proof. Since $v'(y) < 0$ $(y_0 < y < \infty)$, $v'(\infty) = 0$, and for some $y^* > y_0$ the function $v''(y)$ $(y \geq y^*)$ has no zeros, it follows that $v''(y) > 0$ $(y \geq y^*)$. It remains to prove that the set of all zeros of the function $v'''(y)$ is bounded above. Suppose this is not so; then, at points where $v'''(y) = 0$, as a result of differentiation we obtain

$$\sqrt{k^2 - v}\, v^{\mathrm{IV}} = v'' \left[\frac{1}{4\sqrt{(k^2-v)^3}}\, {v'}^2 + \frac{1}{2\sqrt{k^2 - v}}\, v'' + 4(\beta - 1) \right]. \qquad (4)$$

If $\beta \geq 1$, then at these points, for $y \geq y^*$, we have $v^{\mathrm{IV}}(y) > 0$, whence a contradiction easily follows.

If $0 \leq \beta < 1$ and $v''(y) \not\to 0$ (as y tends to ∞), choose a number $\varepsilon \in (0, \min\{8k(1-\beta), \overline{\lim}_{y\to\infty}|v''(y)|\})$, and denote by F the closed set of points on the ray $[y^*, \infty)$ at which $v''(y) \geq \varepsilon$, and by E its complement. Since $\int^{\infty} v''(y)\,dy < \infty$, the set E (as also F) is unbounded. Take any of the intervals of which E consists. At its endpoints $v''(y) = \varepsilon$, and inside it $v''(y) < \varepsilon$. Hence, on this interval, at some point η, a minimum of $v''(y)$ is attained. Take a sequence of points η_i for which $\lim \eta_i = \infty$; for sufficiently large i, at the points η_i,

$$\frac{1}{2\sqrt{k^2-v}}\,v'' + \frac{1}{4\sqrt{(k^2-v)^3}}\,{v'}^2 < \frac{\varepsilon}{2k} + o(1) < 4(1-\beta),$$

and from (4) we obtain $v^{\mathrm{IV}}(\eta_i) < 0$. But this contradicts the fact that at the points η_i a minimum of $v''(y)$ is attained. Lemma 4 is proved.

Corollary.

$$\lim_{y\to\infty} yv'(y) = 0.$$

This relation follows immediately from equation (2) and the boundary conditions.

Lemma 5. If $\beta > 0$, then

$$\lim_{y\to\infty} \frac{yv'(y)}{v(y)} = -\infty.$$

Proof. Let $yv'v^{-1} = z$. Then, with the aid of (2), we obtain

$$z = \frac{4\beta y}{\sqrt{k^2-v}} + z\left(\frac{1}{y} - \frac{2y}{\sqrt{k^2-v}} - \frac{z}{y}\right). \tag{5}$$

If for some $y > y_0$ we have $z'(y) = 0$, then for this y

$$z'' = 4\beta\frac{2(k^2-v)+yv'}{2(k^2-v)^{3/2}} + z\left(-\frac{1}{y^2} - \frac{2(k^2-v)+yv'}{(k^2-v)^{3/2}} + \frac{z}{y^2}\right).$$

In view of the corollary to Lemma 4 and the fact that $z < 0$ ($y \geq y_0$), for sufficiently large y, at the stationary points of $z(y)$ we shall have $z''(y) > 0$. Hence it follows that $z(y)$, beginning with some y, is a monotone function.

Suppose that $z(\infty) = c > -\infty$ ($c \leq 0$). From relation (5) we obtain $z'(\infty) = \infty$, which is impossible. Thus $z(\infty) = -\infty$, as was required to prove.

Theorem. The solution of the boundary-value problem (1) and its first and second derivatives have, as $t \to \infty$, the following asymptotic representation:

$$y(t) = kt - C + t^{-2-2\beta+o(1)}e^{-kt^2+2Ct},$$

$$y'(t) = k - t^{-1-2\beta+o(1)}e^{-kt^2+2Ct},$$

$$y''(t) = t^{-2\beta+o(1)} e^{-kt^2+2Ct},$$

where $C > 0$ is a certain constant (equal to $\lim_{t\to\infty}(kt - y(t))$).

Proof. From equation (2) we obtain

$$\frac{v''(y)}{v'(y)} = -\frac{2y}{\sqrt{k^2 - v}} + \frac{4\beta v}{v'\sqrt{k^2 - v}} = -\frac{2y}{k} + 4yu(y),$$

where, by Lemma 5, $u(y) \to 0$ as $y \to \infty$. Hence, integrating over the interval from some $y_1 > y_0$, we obtain

$$v'(y) = -C_1 e^{-\frac{y^2}{k} + 4\int_{y_1}^{y} su(s)\,ds} \equiv -C_1\chi(y) \qquad (C_1 > 0;\ y > y_0) \tag{6}$$

and, consequently,

$$\lim_{y\to\infty} \frac{yv}{v'} = \lim_{y\to\infty} \left(-\frac{\displaystyle\int_y^\infty \chi(s)\,ds}{y^{-1}\chi(y)} \right) = -\frac{k}{2} \tag{7}$$

(the limit is easily found by l' Hospital' s rule),

$$v = \frac{C_2 + o(1)}{y}\chi(y) \qquad \left(y_0 < y < \infty;\ C_2 = \frac{k}{2}C_1\right). \tag{8}$$

Next, by l' Hospital' s rule we have

$$\lim_{y\to\infty} y^2\left(k - \sqrt{k^2 - v}\right) = \lim_{y\to\infty} \frac{v'/2\sqrt{k^2 - v}}{-2y^{-3}} = \lim_{y\to\infty} \frac{-C_1 y^3\chi(y)}{-4\sqrt{k^2 - v}} = 0,$$

whence, in view of (7), $\lim_{y\to\infty} y^2 u(y) = -\frac{\beta}{2}$. Therefore, by l' Hospital' s rule,

$$\lim_{y\to\infty} \frac{\displaystyle\int_{y_1}^{y} su(s)\,ds}{\ln y} = \lim_{y\to\infty} y^2 u(y) = -\frac{\beta}{2}. \tag{9}$$

Therefore, by virtue of (8),

$$\sqrt{k^2 - v} = k - y^{-1-2\beta+o(1)} e^{-\frac{y^2}{k}}. \tag{10}$$

Thus, first of all, since $\sqrt{k^2 - v} = y'(t)$, we have $y(t) \sim kt$, and on the basis of (10)

$$y'(t) = k - e^{-kt^2 + o(t^2)} \qquad (t \geq t_0).$$

Integrating, we obtain

$$y = kt - C + \int_t^\infty e^{-k\tau^2 + o(\tau^2)}\, d\tau \qquad \left(C = kt_0 - y_0 + \int_{t_0}^\infty e^{-k\tau^2 + o(\tau^2)}\, d\tau \right).$$

Put, for $t \geq t_0$,

$$\int_t^\infty e^{-k\tau^2 + o(\tau^2)}\, d\tau = e^{-kt^2 + \gamma(t)t^2}.$$

Then it is easy to show that $\gamma(\infty) = 0$. Indeed, by l' Hospital' s rule,

$$\frac{\int_t^\infty e^{-\alpha\tau^2}\, d\tau}{e^{-\alpha t^2} t^{-1}} \underset{t\to\infty}{\longrightarrow} \frac{1}{2\alpha} \qquad (\alpha > 0);$$

but for any $\varepsilon > 0$ $(\varepsilon < k)$, for sufficiently large t,

$$e^{-(k+\varepsilon)t^2} < e^{-kt^2 + o(t^2)} < e^{-(k-\varepsilon)t^2},$$

and therefore

$$t^{-1} e^{-(k+\varepsilon)t^2} \left(\frac{1}{2(k+\varepsilon)} + o(1) \right) < \int_t^\infty e^{-k\tau^2 + o(\tau^2)}\, d\tau < t^{-1} e^{-(k-\varepsilon)t^2} \left(\frac{1}{2(k-\varepsilon)} + o(1) \right),$$

whence our assertion follows at once. Thus,

$$y = kt - C + e^{-kt^2 + o(t^2)}. \tag{11}$$

Hence $C = \lim_{t\to\infty}(kt - y)$; from Lemma 1 it follows that the difference $kt - y(t)$, for $0 \leq t < \infty$, increases, and, since $y(0) = 0$, $C > 0$.

Substituting (11) into (10), we obtain the expression for $y'(t)$ given in the formulation of the theorem. Further, for $t > t_0$, according to (6),

$$y''(t) = \frac{-v'(y)}{2} = \frac{C_1}{2} e^{-\frac{y^2}{k} + 4\int_{y_1}^{y} su(s)\, ds} = y^{-2\beta + o(1)} e^{-\frac{y^2}{k}}$$

(see (9)). Using (11), we obtain the required expression for $y''(t)$.

Finally, integrating the expression for $y'(t)$ and taking into account that

$$\int_t^\infty \tau^{-1-2\beta+o(1)} e^{-k\tau^2+C\tau}\, d\tau = t^{-2-2\beta+o(1)} e^{-kt^2+2Ct}$$

(the proof is analogous to the proof that $\gamma(\infty) = 0$), and then comparing the result obtained with (11), we obtain the required representation of $y(t)$.

Let us note in conclusion that the value of the constant C is unknown. It would be important to obtain an estimate, or a method for approximate computation, or an expansion in a series of some kind, etc., for this constant.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.